Construction of very high order Residual Distribution Schemes for steady problems

Rémi Abgrall[⋆], Mario Ricchiuto[⋆], Cédric Tavé[⋆], Nadège Villedieu[†] and Herman Deconinck[†]

⋆ Mathématiques Appliquées de Bordeaux, Projet Scalapplix, INRIA FutURs, Talence, France

† von Karman Institute for Fluid Dynamics, Rhodes St Genese, Belgium

Overview

1. RDS schemes
2. Construction on scalar advection
3. Numerical examples
    - Scalar
    - Systems

Forewords

- This lecture foccusses on steady problems.
- Several JCP papers on second order unsteady.
- Existing work on coupling hyperbolic/convective terms to viscous one (Villedieu–Ricchiuto)

Discussion on scalar problems.

$$\begin{aligned} \vec{\lambda}\cdot\nabla u &= 0 \quad && x\in\Omega \\ u &= g && x\in\Gamma^- \end{aligned}$$

- unstructured triangular meshes, triangles T,
- $u_\sigma \simeq u(M_\sigma)$, σ d.o.f
- scheme $u_\sigma^{n+1} = u_\sigma^n - \omega_i \sum\limits_{T,\sigma\in T} \Phi_\sigma^T$.
- steady solution i.e. $n \rightarrow +\infty$.

Second order

- σ : vertices of the triangles
- u^h piecewise linear interpolant of $\{u_\sigma\}$,
- Conservation relation

$$\begin{aligned} \sum_{\sigma\in T} \Phi_\sigma &= \int_T \vec{\lambda}\cdot\nabla u^h dx := \Phi^T \\ &= \sum_{\sigma\in T} k_\sigma^T u_\sigma \qquad k_\sigma = \int_T \vec{\lambda}\cdot\nabla N_\sigma \end{aligned}$$

Example : piecewise interpolation

- Upwind finite volume scheme,
- N scheme

$$\Phi_\sigma = k_\sigma^+ (u_\sigma - \widetilde{u})$$

$$\widetilde{u} = \left(\sum_{\sigma'} k_{\sigma'}^- \right)^{-1} \left(\sum_{\sigma' \in T} k_{\sigma'}^- u_{\sigma'} \right)$$

- Local Rusanov,

$$\Phi_\sigma = \frac{1}{3} \left(\Phi^T + \alpha \sum_{\sigma' \neq \sigma, \sigma \in T} (u_\sigma - u_{\sigma'}) \right)$$

Example : piecewise interpolation

The upwind FV scheme, N scheme and Rusanov scheme are monotone schemes

$$\Phi_\sigma = \sum_{\sigma' \neq \sigma} c_{\sigma\sigma'}(u_\sigma - u_{\sigma'}), \qquad c_{\sigma\sigma'} \geq 0.$$

Provides a local maximum principle on the solution

Summary

- Conservation property

$$\sum_{\sigma\in T}\Phi_\sigma=\int_T\vec{\lambda}\cdot\nabla u^h dx:=\Phi^T$$

- Second order approximation of the flux (here because $u^h=u+\mathcal{O}(h^2)$.
- LP condition ($\Phi_\sigma^T(u)=O(h^3)$ for the exact solution)
- Monotonicity preserving scheme ($\Phi_\sigma=\sum_{\sigma'}c_{\sigma\sigma'}(u_\sigma-u_{\sigma'})$)
- technique for going from first order to second order.

$$\text{scheme : }\sum_{T,\sigma\in T}\Phi_\sigma^T=0$$

How to get really

high order, compact, monotonicity preserving schemes on general meshes ?

- Use of P^k interpolation,
- “Improved” conservation relation
- Probably more general than this (Hermite, spectral, etc)

Other contributions

- Abgrall-Roe, J. Scientific Computing, 2001
- Hubbard (Computer and Fluids, 2005)
- Ricchiuto et al (VKI LS, 2005)
- Andrianov et al. (IJNM, 2005)
- De Palma et al. (JCP in press)
- . . .

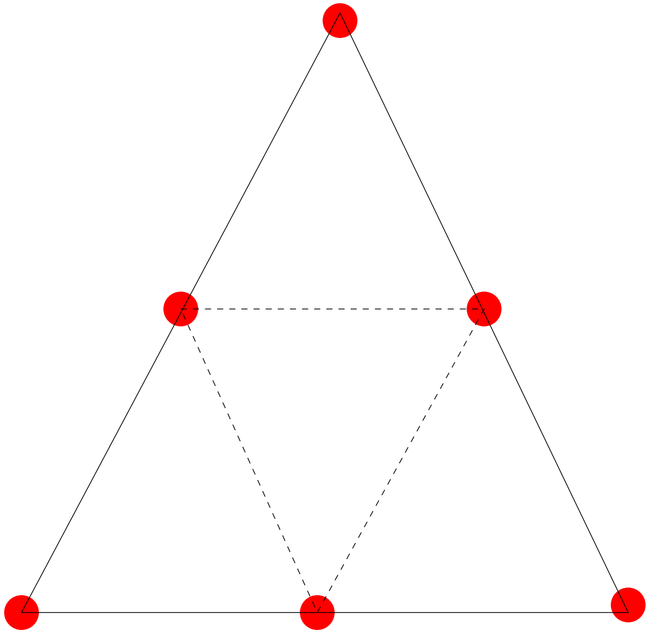
Notations–degrees of freedom

- mesh τ, triangles T, verticesM_j,
- we seek a solution that is piecewise polynomial of degree k in each triangle,
- need to provide $(k+1)(k+2)/2$ degrees of freedom :

Scheme

$$\sum_{\sigma\in T} \Phi_\sigma^T = 0$$

Example, $k = 2$



Degree of freedom P^2 interpolation.

Structural condition :

Lax Wendroff like result+ high order

- $\Phi^T := \int_T \text{div}\, F^h(u^h)dx = \int_T \vec{\lambda}\cdot\nabla u^h dx = \sum_{\sigma\in T} \Phi_\sigma^T(u^h),$

  + standard assumption Lax Wendroff
  solution of

  $$\text{div}\,\mathbf{F}(u) = 0 \qquad + \text{boundary conditions}$$

- High order if $\Phi_\sigma^T(u^h) = O(h^{2+k})$ for smooth solutions+ regular meshes

Monotonicity condition

Ensures stability in L^∞

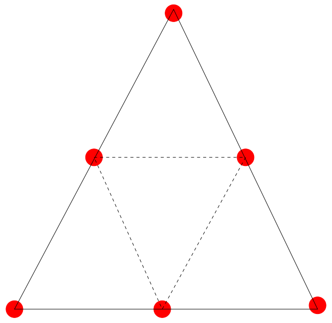
$$\Phi_\sigma = \sum_{\sigma' \neq \sigma, \sigma' \in T} c_{\sigma\sigma'} (u_\sigma - u_{\sigma'})$$

with

$$c_{\sigma\sigma'} \geq 0$$

dependant on the solution.

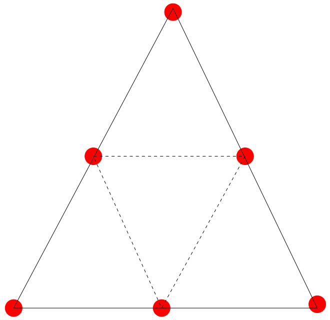
Previous try (*Abgrall-Roe, JSC, 2001*)



- define sub-triangulation
- use a “reference” monotone first order scheme $\Phi_\sigma^{L,T'}$ in each sub–triangulation
- define sub–residuals in each sub-triangle accordingly
- $$u_\sigma^{n+1}=u_\sigma^n-\omega_i\sum_{T,\sigma\in T}\Psi_\sigma^T,\qquad \Psi_\sigma^T=\sum_{T'_T\subset T,\sigma\in T'_T}\left(\Phi_\sigma^{T'_T}\right)^\star$$

First try

- We can construct the N scheme on the sub–triangles of T : $\Phi_\sigma^{T_\xi}$ for $\xi = I, II, III, IV$



- Consider

$$\Phi_\xi = \int_{T_\xi} \vec{\lambda} \cdot \nabla u^h. \qquad \text{quadratic}$$

- clear that linear $\Phi_\xi \neq \sum_{\sigma \in T_\xi} \Phi_\sigma^{T_\xi}$
- but we still can use the N scheme for a comparison purpose and want .

$$\Psi_\sigma^{T_\xi} = \beta_\sigma^\xi \Phi_\xi$$

First try

Main problem :

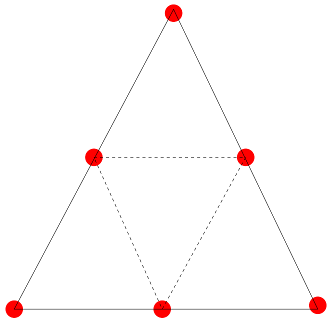
$$\sum_{\sigma\in T'} \Phi_\sigma^{L,T} \neq \int_{T'} \vec{\lambda}\cdot\nabla u^h = \sum_{\sigma'\in T'} \left(\Phi_\sigma^{T'_T}\right)^\star$$

Problem is not conservation but algebraic
Construction of $\beta_\sigma^{T'}$ needs this condition.

An always defined solution

- Start (for example) from the Rusanov scheme
- “limit” residuals
- update

Rusanov scheme



$$\Phi_\sigma^T = \frac{1}{6}\Big(\int_{\partial T}[\vec{\lambda}\cdot\vec{n}]u^h dx + \alpha_T \sum_{\sigma'\in T}(u_{\sigma'} - u_\sigma)\Big)$$

with

$$\alpha_T \geq \max_{\sigma\in T}\Big|\int_T \vec{\lambda}\cdot\nabla\mathcal{N}_\sigma dx\Big|.$$

monotone scheme.

limiting procedure

For Rusanov scheme,

$$\sum_{\sigma \in T} \Phi_\sigma^T = \int_{\partial T} [\vec{\lambda} \cdot \vec{n}] u^h dx = \Phi^T.$$

- define $x_\sigma = \Phi_\sigma^R / \Phi^T$,
- set $\beta_\sigma^T = \dfrac{x_\sigma^+}{\sum\limits_{\sigma'} x_{\sigma'}^+}$ and $\Phi_\sigma^T := \beta_\sigma^T \Phi^T$.

always defined because $\sum\limits_{\sigma'} \Phi_\sigma^{H,T} = \Phi^T$ implies $\sum\limits_{\sigma'} x_{\sigma'} = 1$ so that $\sum\limits_{\sigma'} x_{\sigma'}^+ \geq 1$

.

Properties

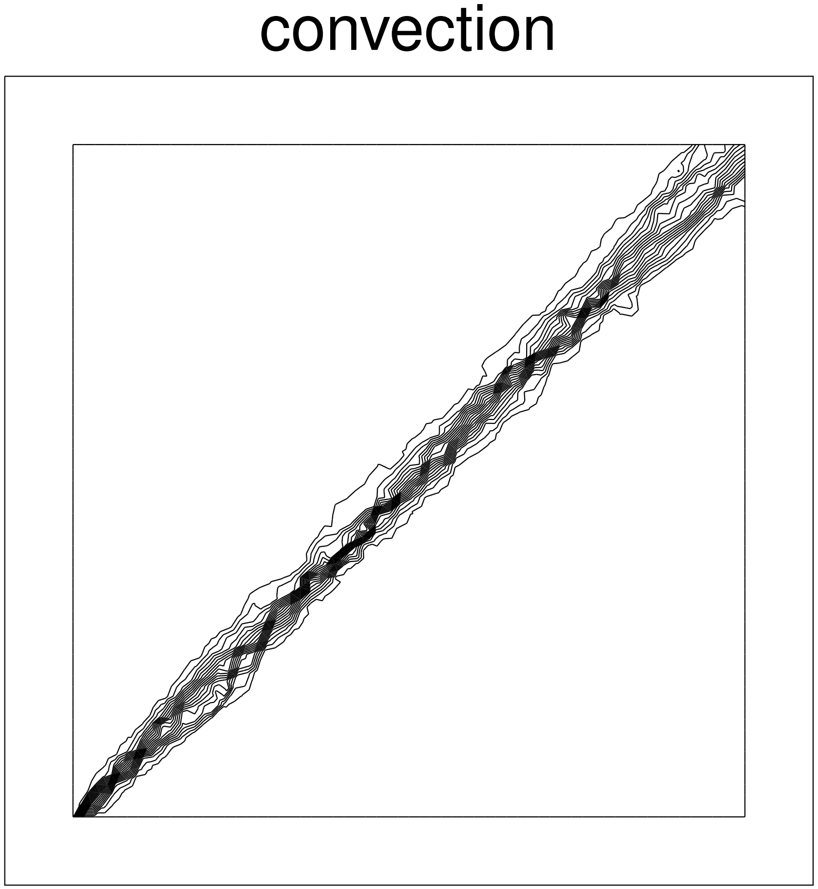
If there exists a unique solution to

$$\begin{array}{lll} \displaystyle\sum_{T\ni\sigma} \Phi_\sigma^{H,T} = 0 & \sigma \in \Omega & (\vec{\lambda}\cdot\nabla u = 0) \\ u_\sigma = g & \text{on } \Gamma^- & \end{array}$$

then the scheme is third order accurate because

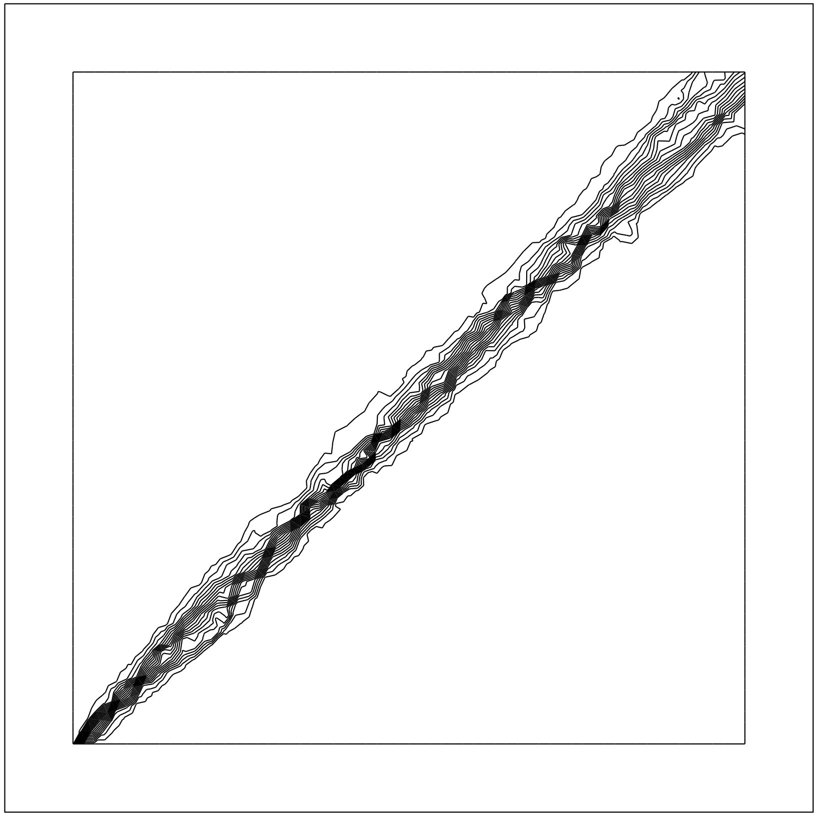
$$\int_{\partial T} \vec{\lambda}\cdot\nabla u^h dl = O(h^{3+1})$$

Numerical experiments

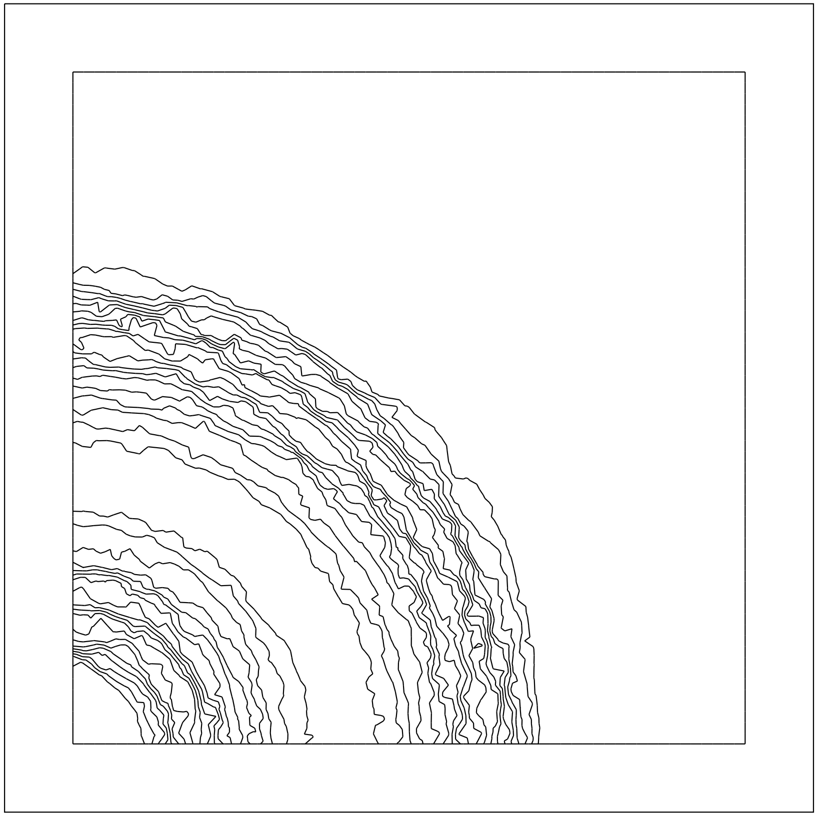


Numerical experiments

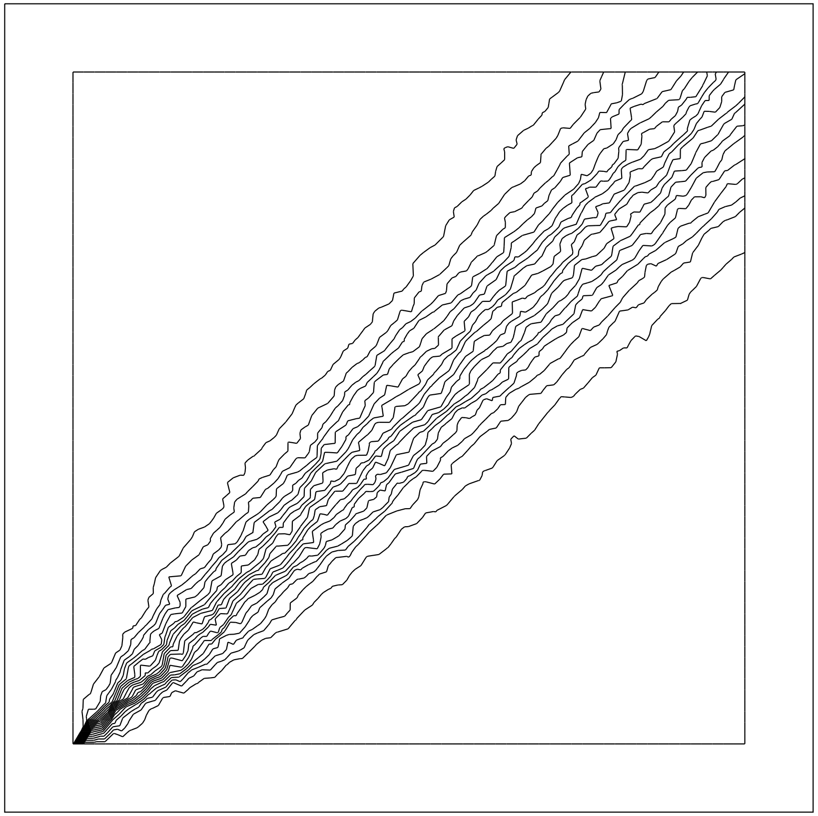
convection



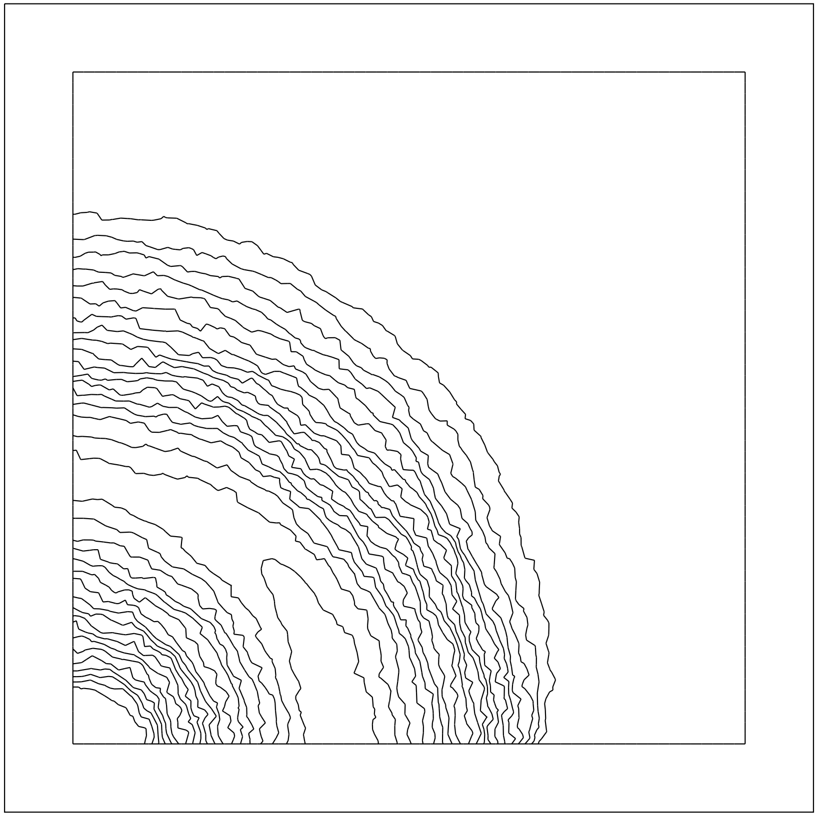
solid rotation



Same second order



convection



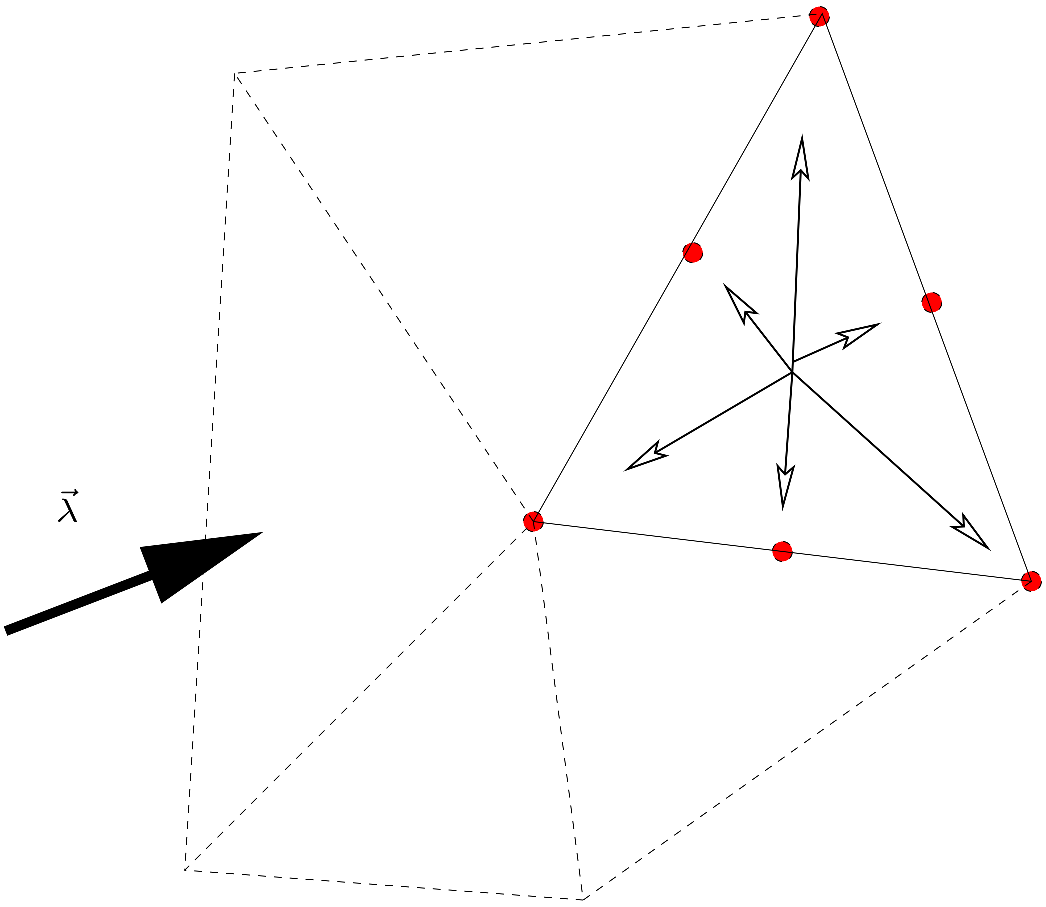
solid rotation

Why : existence of mild spurious mode

- Analysis : see Abgrall, Essentially non oscillatory Residual Distribution schemes for hyperbolic problems, JCP, 2006.
- Interpretation

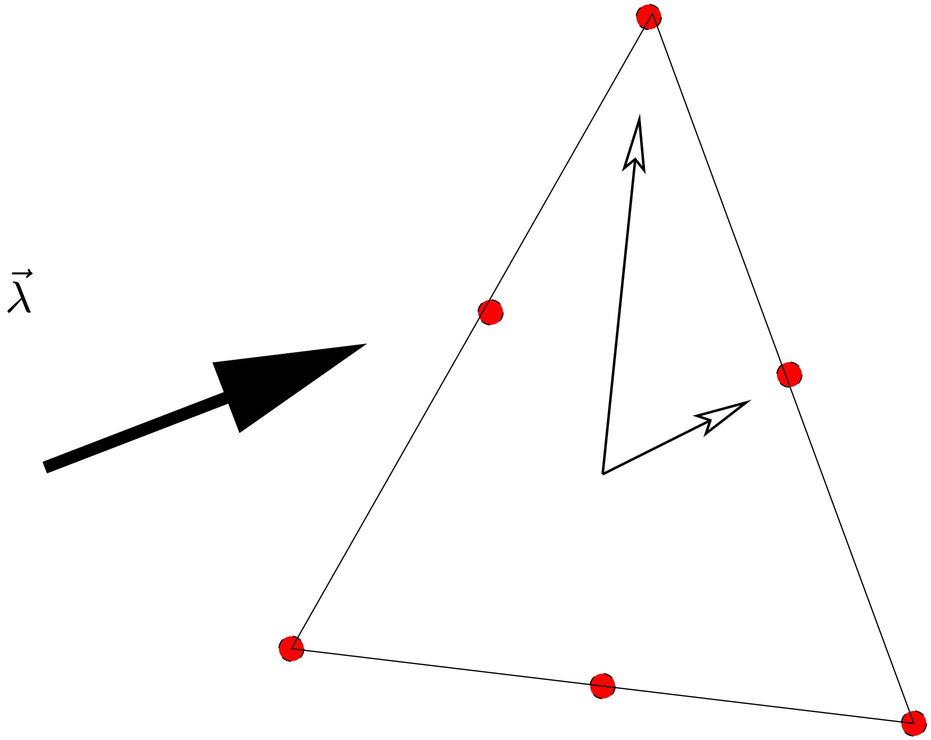
Interpretation

First order scheme



Interpretation

$$\Phi_\sigma^\star = \beta_\sigma \Phi^T$$



This problem does not exist for genuinely upwind schemes

Fix

Force some upwinding : add

$$\Theta(u^h)h\int_T\left(\vec{\lambda}\cdot\nabla\mathcal{N}_\sigma\right)\left(\vec{\lambda}\cdot\nabla u^h\right)$$

Fix

Scheme

$$\sum_{T\ni\sigma}\left(\Phi_\sigma^{H,T}\right)^\star = 0 \quad \sigma\in\Omega \qquad (\vec\lambda\cdot\nabla u=0)$$

$$u_\sigma = g \qquad \text{on } \Gamma^-$$

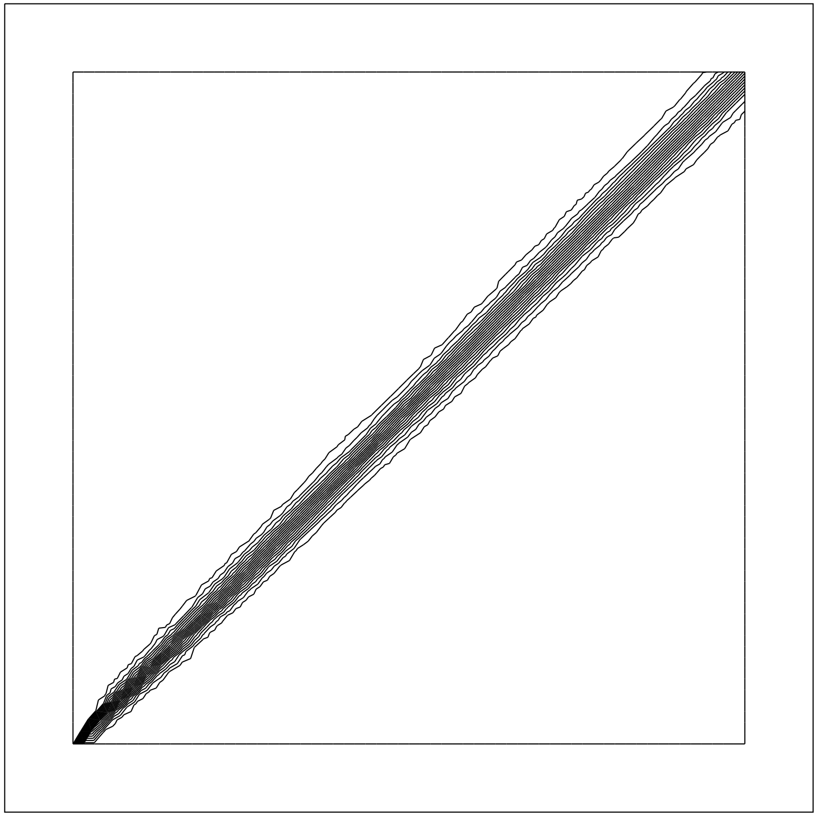
with

$$\left(\Phi_\sigma^{H,T}\right)^\star = \beta_\sigma^T\left(\int_T \vec\lambda\cdot\nabla u dx\right) + \Theta(u^h)h\int_T\left(\vec\lambda\cdot\nabla\mathcal{N}_\sigma\right)\left(\vec\lambda\cdot\nabla u^h\right)$$

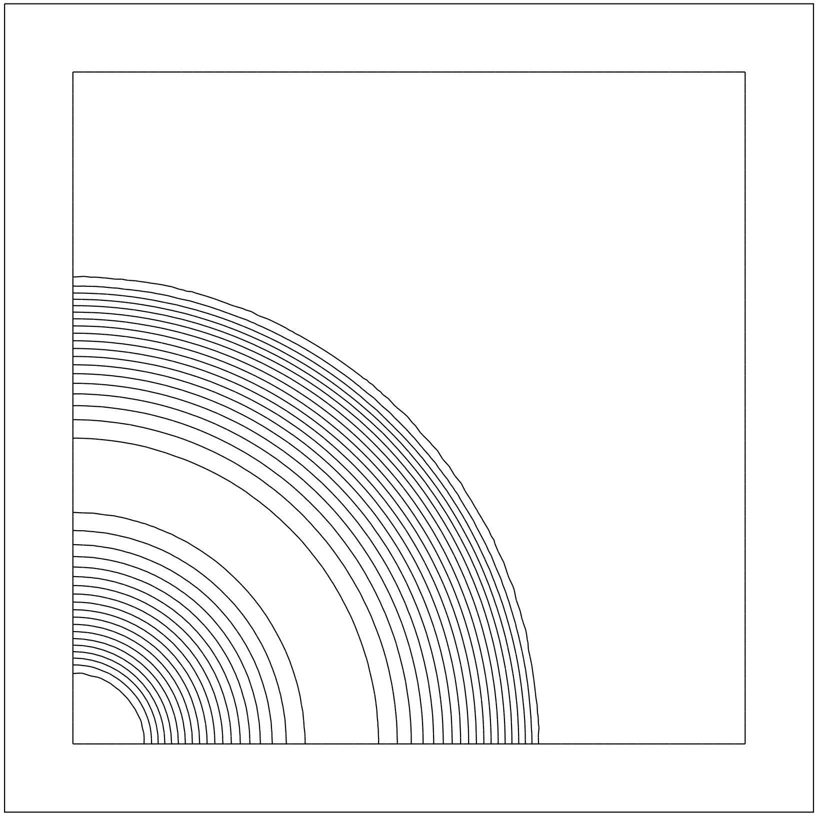
Still third order accurate, not any more (formaly) monotonicity preserving

but essentially non oscillatory

Numerical experiments



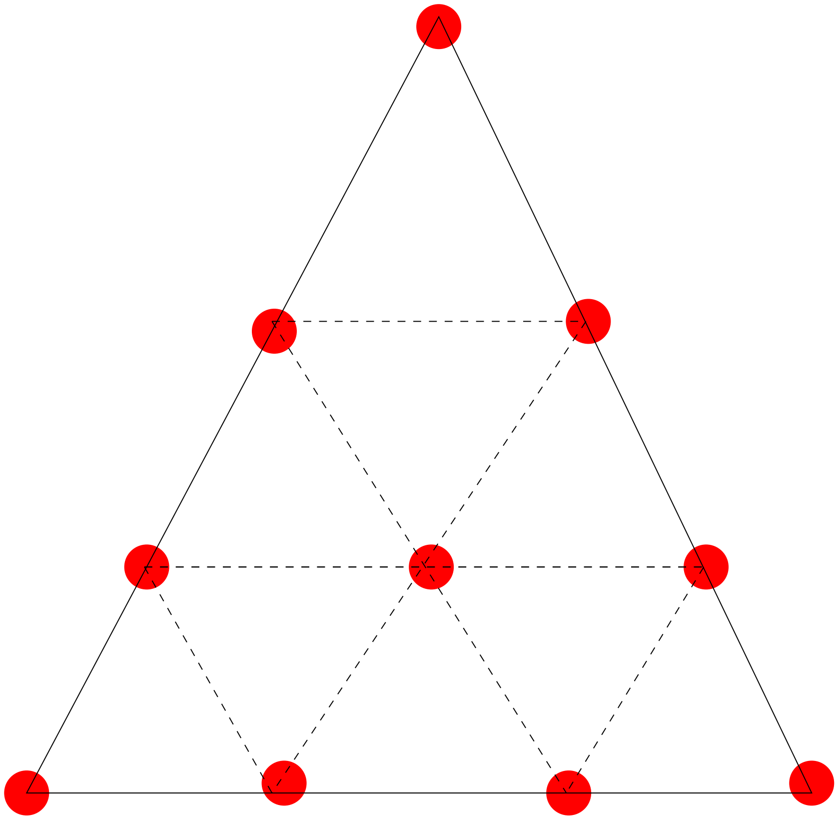
convection



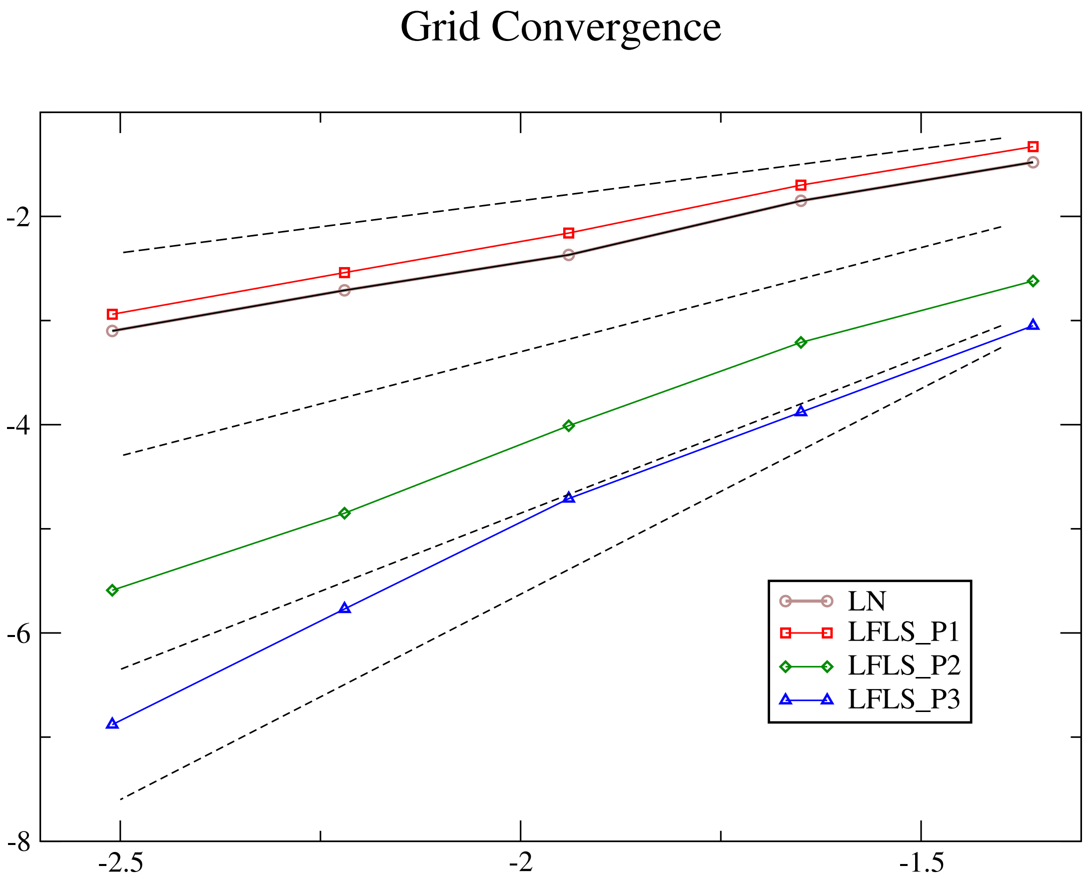
solid rotation

Fourth order

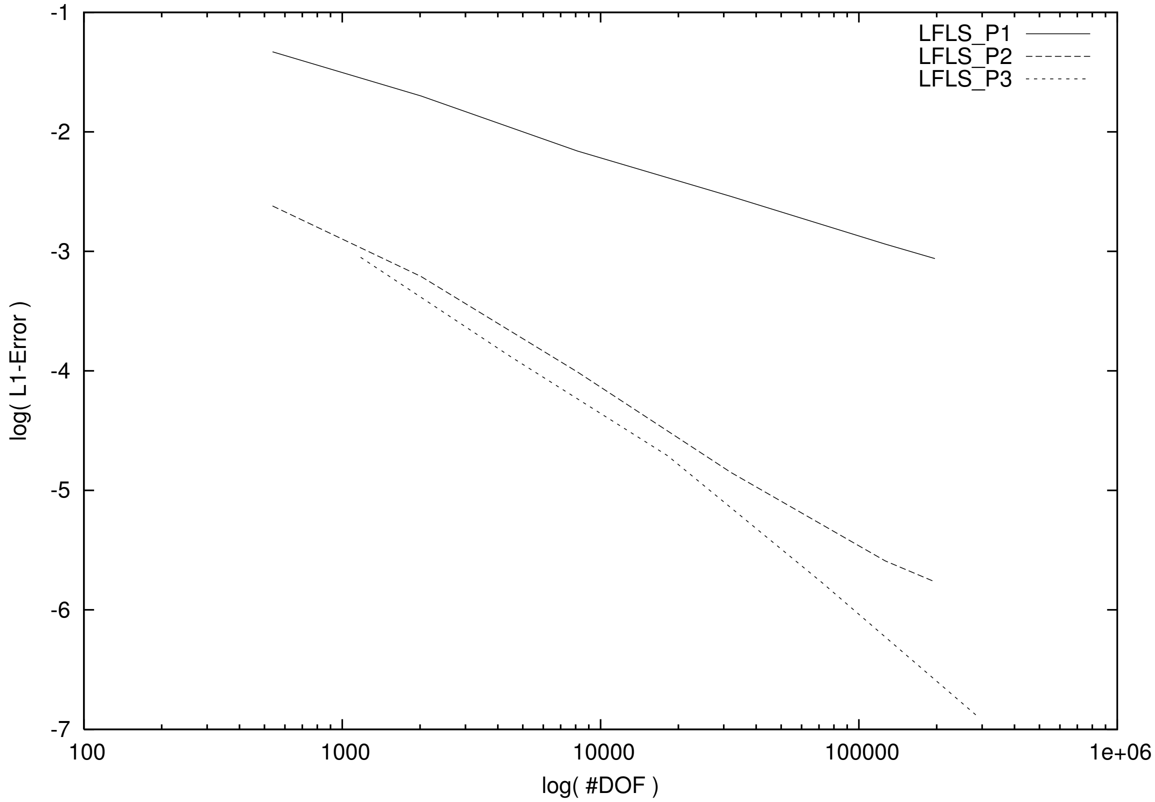
Same. Replace quadratic element by cubic elements (10 dof/element)



Accuracy : convection

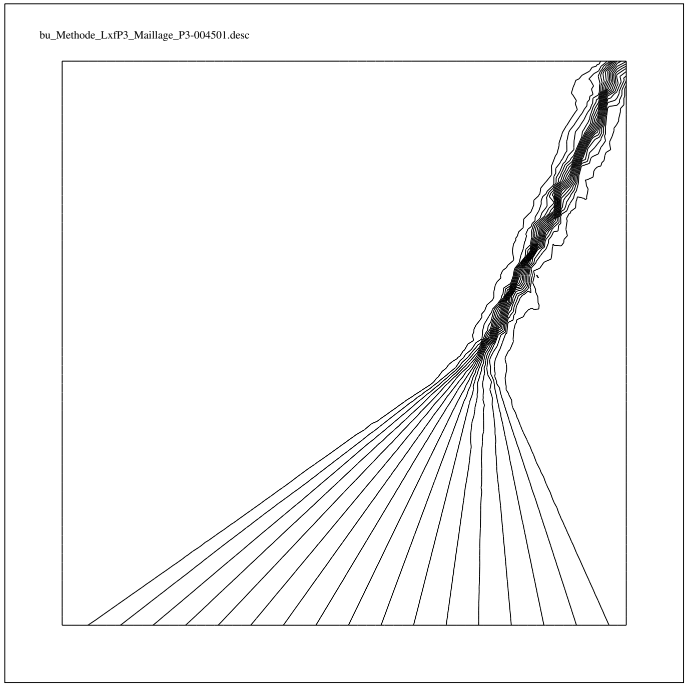


Accuracy/ # of dof



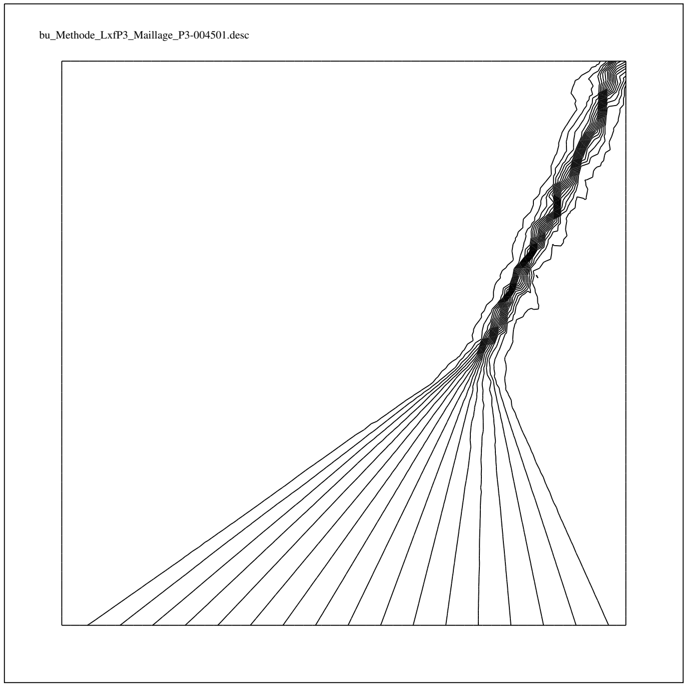
Burgers equation (4th order)

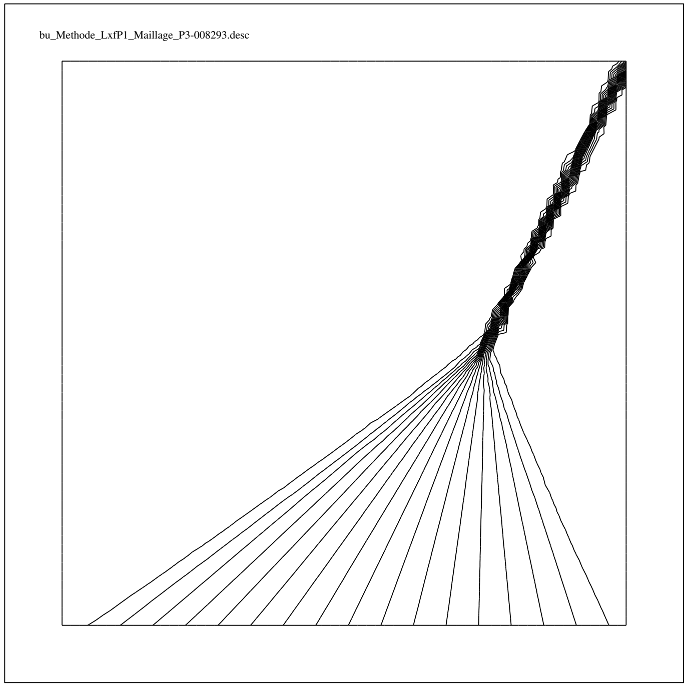
$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0, \qquad u(x,y) = 1.5 - 2x \text{ on inflow boundaries}$$



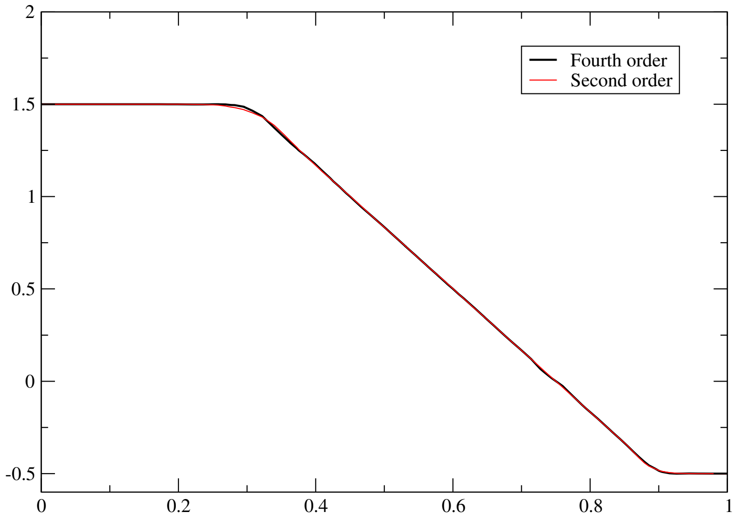
Burgers equation (2nd/3rd order)

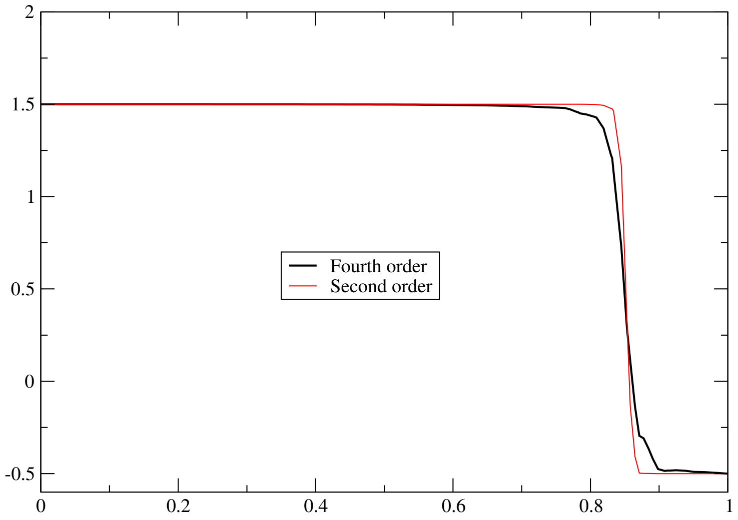
same number of dof.



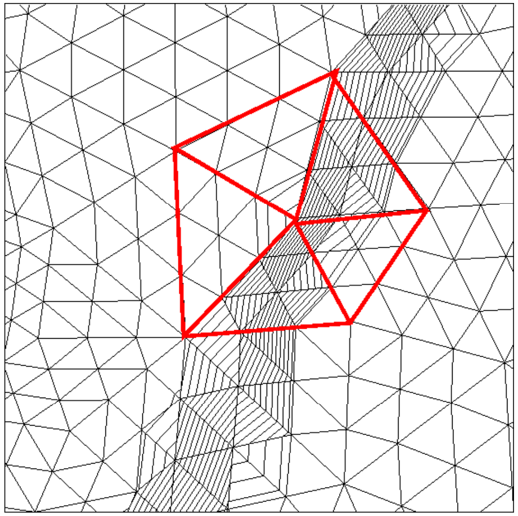


Comparison, 1

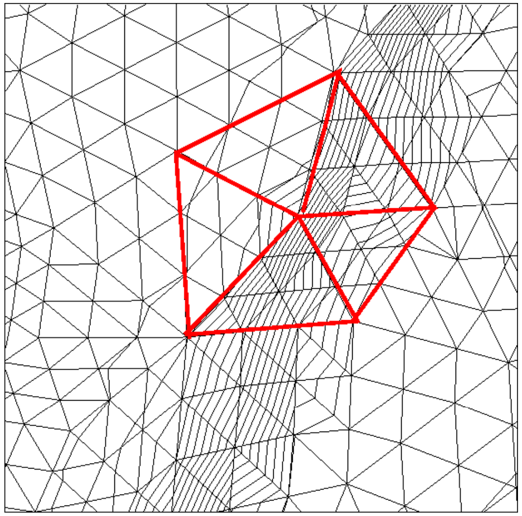




Comparison, 2



2nd order

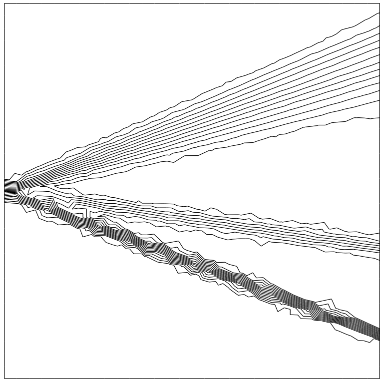


4th order

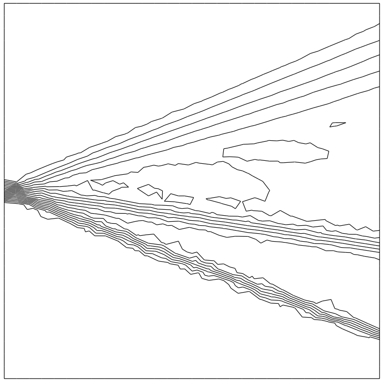
Fluid Mechanics examples

- Stabilisation procedure : on characteristic waves. Similar as Abgrall-Mezine, Construction of second-order accurate monotone and stable residual distribution schemes for steady problems. J. Comput. Phys., 195(2):474–507, 2004.
- additionnal stabilisation : formal extension to systems

Jet (3rd order)



ρ



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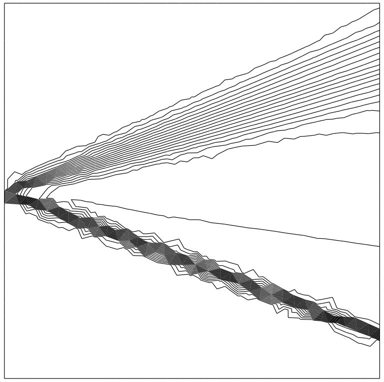
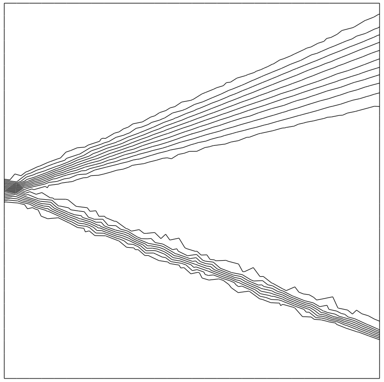


v



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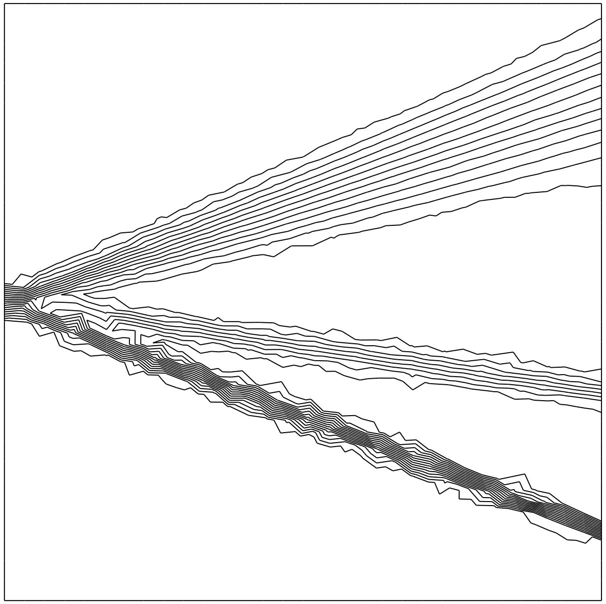
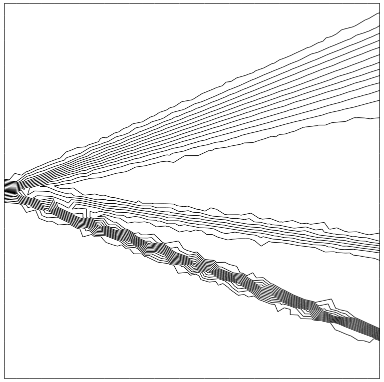
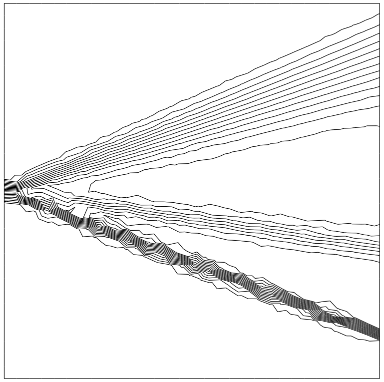
Jet (3rd order) without dissipation



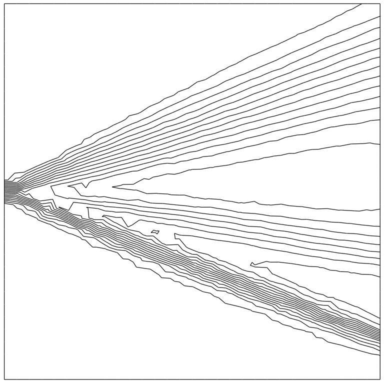
O1/O2/O3, same dof



O3

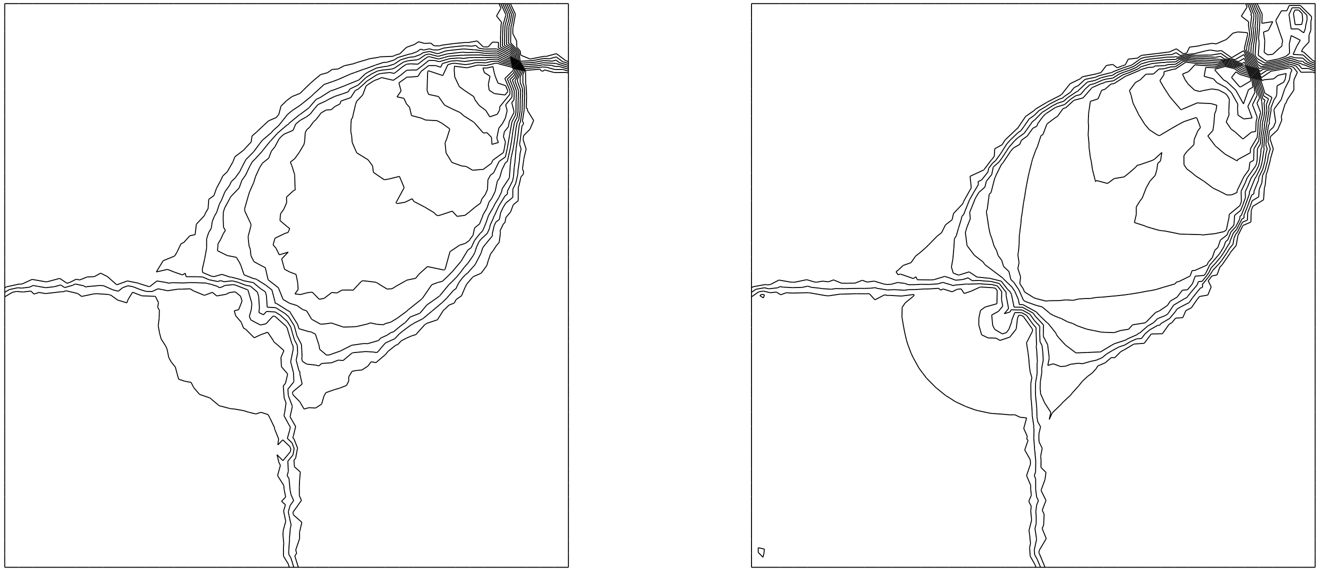


O2

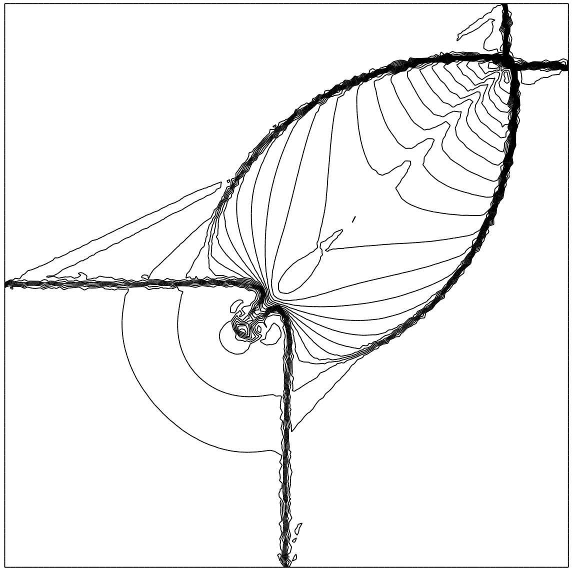


O1

4 state shock tube



4 state shock tube



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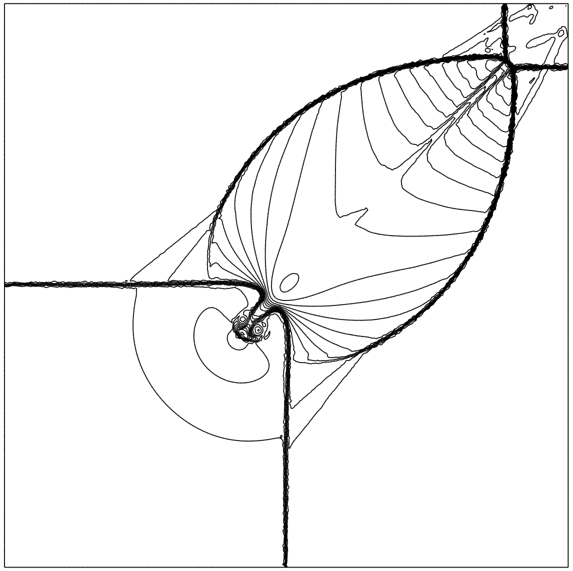

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Conclusions, perpectives.

- Residual distribution of high order (3rd, 4th),
- Essentially non oscillatory,
- Preliminary results for fluid mechanics

To be done

- Avoid additional stabilisation ? (system)
- Boundary treatement
- Efficiency
- Unsteady

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ADIGMA !