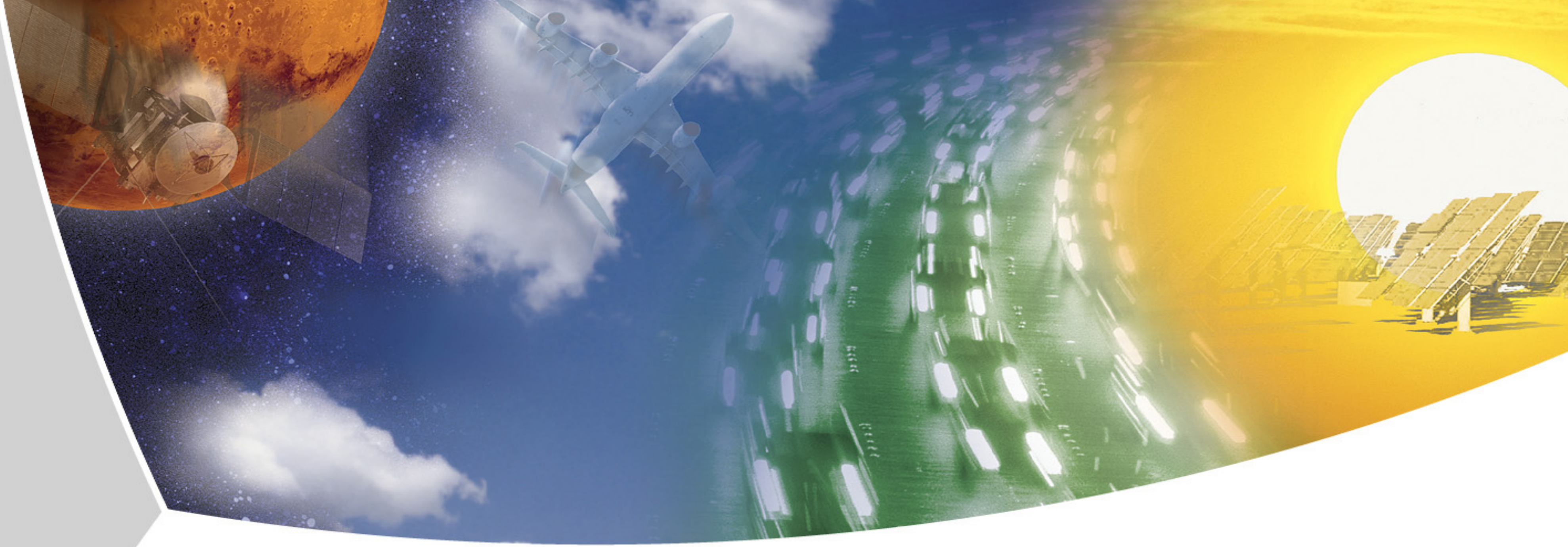

Error estimation and adjoint-based adaptation in aerodynamics

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Overview

- DG discretization of the compressible Euler and Navier-Stokes equations
- Residual-based indicators for adaptive mesh refinement
- Error estimation and adjoint-based indicators for goal-oriented refinement
- Adjoint consistent DG discretizations

The compr. Euler and Navier-Stokes equations in 2D

The compressible Euler equations:

$$\frac{\partial}{\partial t}\begin{pmatrix}\varrho \\ \varrho v_1 \\ \varrho v_2 \\ \varrho E\end{pmatrix} + \frac{\partial}{\partial x_1}\begin{pmatrix}\varrho v_1 \\ \varrho v_1^2 + p \\ \varrho v_1 v_2 \\ v_1(\varrho E + p)\end{pmatrix} + \frac{\partial}{\partial x_2}\begin{pmatrix}\varrho v_2 \\ \varrho v_1 v_2 \\ \varrho v_2^2 + p \\ v_2(\varrho E + p)\end{pmatrix} = 0$$

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The compressible Navier-Stokes equations:

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot \mathcal{F}^c(\mathbf{u}) - \nabla \cdot \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) = 0$$

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The compressible Navier-Stokes equations:

$$\frac{\partial}{\partial t}\mathbf{u}+\nabla\cdot\mathcal{F}^c(\mathbf{u})-\nabla\cdot\mathcal{F}^v(\mathbf{u},\nabla\mathbf{u})=0$$

$$\mathbf{f}_1^v(\mathbf{u},\nabla\mathbf{u})=\begin{pmatrix}0\\ \tau_{11}\\ \tau_{21}\\ \tau_{11}v_1+\tau_{12}v_2+\kappa T_{x_1}\end{pmatrix},\quad \mathbf{f}_2^v(\mathbf{u},\nabla\mathbf{u})=\begin{pmatrix}0\\ \tau_{12}\\ \tau_{22}\\ \tau_{21}v_1+\tau_{22}v_2+\kappa T_{x_2}\end{pmatrix}.$$

DG discretization of the compr. Euler equations

The problem:

$$\nabla \cdot \mathcal{F}^c(\mathbf{u}) = 0 \quad \textbf{in } \Omega,$$

with $\mathbf{u} = (\varrho, \varrho v_1, \varrho v_2, \rho E)^T$.

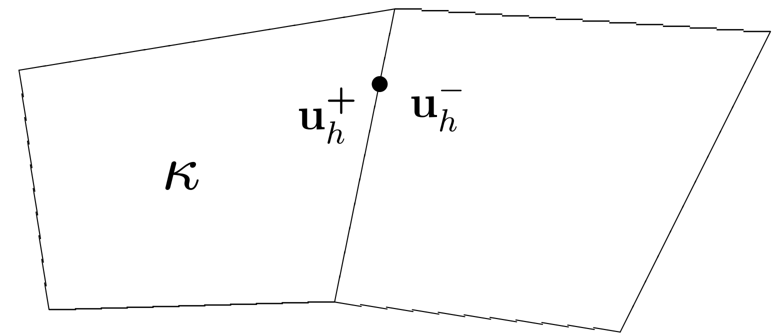
The discretization of DG(p): Find u_h in $\mathrm{V}_{h,p}$ such that

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) \equiv \sum_{\kappa \in \mathcal{T}_h} \left\{ - \int_\kappa \mathcal{F}^c(\mathbf{u}_h) : \nabla \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \int_{\partial\kappa \setminus \Gamma} \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}_\kappa) \, \mathbf{v}_h^+ \, \mathrm{d}s \right\}$$
$$+ \int_\Gamma \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}_\kappa) \, \mathbf{v}_h^+ \, \mathrm{d}s = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_{h,p},$$

with

$$\mathbf{V}_{h,p} = \left\{ \mathbf{v} \in [L_2(\Omega)]^4 : \mathbf{v}|_\kappa \in [\mathcal{Q}_p(\kappa)]^4 \;\; \forall \kappa \in \mathcal{T}_h \right\},$$

DG discretization of the compr. Euler equations



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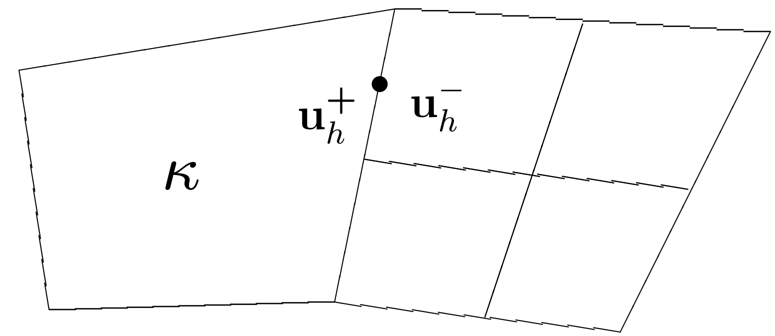

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DG discretization of the compr. Navier-Stokes equations

Find u_h in $\mathrm{V}_{h,p}$ such that

$$\mathcal{N}(\mathbf{u}_h,\mathbf{v}_h) \equiv -\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \mathcal{H}(\mathbf{u}_h^+,\mathbf{u}_h^-,\mathbf{n}_\kappa)\cdot\mathbf{v}_h^+ \,\mathrm{d}s$$
$$+\int_\Omega \mathcal{F}^v(\mathbf{u}_h,\nabla_h\mathbf{u}_h) : \nabla_h\mathbf{v}_h \,\mathrm{d}\mathbf{x} - \int_{\Gamma_\mathcal{I}} \{\!\{G(\mathbf{u}_h)\nabla\mathbf{u}_h\}\!\} : \underline{[\![\mathbf{v}_h]\!]} \,\mathrm{d}s$$
$$-\int_{\Gamma_\mathcal{I}} \{\!\{G^\top(\mathbf{u}_h)\nabla\mathbf{v}_h\}\!\} : \underline{[\![\mathbf{u}_h]\!]})\,\mathrm{d}s + \int_{\Gamma_\mathcal{I}} \delta\underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s + \mathcal{N}_\Gamma(\mathbf{u}_h,\mathbf{v}_h) = 0$$

for all v_h in $\mathrm{V}_{h,p}$, with

$$\mathcal{N}_\Gamma(\mathbf{u}_h,\mathbf{v}_h) \equiv \int_\Gamma \mathcal{H}_\Gamma(\mathbf{u}_h^+,\mathbf{u}_\Gamma(\mathbf{u}_h^+),\mathbf{n})\cdot\mathbf{v}_h^+\,\mathrm{d}s + \int_\Gamma \delta\left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right)\cdot\mathbf{v}_h^+\,\mathrm{d}s,$$
$$-\int_\Gamma G_\Gamma(\mathbf{u}_h)\nabla\mathbf{u}_h : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s$$
$$-\int_\Gamma G_\Gamma^\top(\mathbf{u}_h)\nabla\mathbf{v}_h : \left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right)\otimes\mathbf{n}\,\mathrm{d}s$$

Adaptive mesh refinement

Refinement indicators:

- Residual-based indicators
- Adjoint-based indicators (for goal-oriented refinement)

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- **Residual-based indicators**
- **Adjoint-based indicators (for goal-oriented refinement)**

1. **Use *residual-based indicators* for resolving all flow features. They are known to be reliable, i.e. it is guaranteed that under local refinement the solution converges to the exact solution**

Adaptive mesh refinement

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- Residual-based indicators
- Adjoint-based indicators (for goal-oriented refinement)

1. Use *residual-based indicators* for resolving all flow features. They are known to be reliable, i.e. it is guaranteed that under local refinement the solution converges to the exact solution

2. For the exact computation of a specific target quantity $J(\cdot)$, like for example
   - the drag coefficient, the lift coefficient, ...

   use the so-called *adjoint-based indicators* for goal-oriented refinement

Residual-based indicators

$$\eta_\kappa^{\text{res}} = \quad \|h_\kappa^s \mathbf{R}(\mathbf{u}_h)\|_{L_2(\kappa)}$$
$$+ \|h_\kappa^{s-1/2}\mathbf{r}(\mathbf{u}_h)\|_{L_2(\partial\kappa\setminus\Gamma)} + \|h_\kappa^{s-3/2}\boldsymbol{\rho}(\mathbf{u}_h)\|_{L_2(\partial\kappa\setminus\Gamma)}$$
$$+ \|h_\kappa^{s-1/2}\mathbf{r}_\Gamma(\mathbf{u}_h)\|_{L_2(\partial\kappa\cap\Gamma)} + \|h_\kappa^{s-3/2}\boldsymbol{\rho}_\Gamma(\mathbf{u}_h)\|_{L_2(\partial\kappa\cap\Gamma)}$$

$$\mathbf{R}(\mathbf{u}_h) = -\nabla\cdot\mathcal{F}^c(\mathbf{u}_h) + \nabla\cdot\mathcal{F}^v(\mathbf{u}_h, \nabla_h\mathbf{u}_h),$$
$$\mathbf{r}(\mathbf{u}_h) = \mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}_h^+) - \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}^+) - \frac{1}{2}[\![\mathcal{F}^v(\mathbf{u}_h, \nabla_h\mathbf{u}_h)]\!] - \delta[\mathbf{u}_h],$$
$$\boldsymbol{\rho}(\mathbf{u}_h) = -\frac{\theta}{2}\left(G(\mathbf{u}_h)\underline{[\![\mathbf{u}_h]\!]}\right)^\top,$$
$$\mathbf{r}_\Gamma(\mathbf{u}_h) = \mathbf{n}\cdot\mathcal{F}^c(\mathbf{u}_h^+) - \mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n})$$
$$- \mathbf{n}\cdot\mathcal{F}^v(\mathbf{u}_h^+, \nabla_h\mathbf{u}_h^+) + \mathbf{n}\cdot\mathcal{F}_\Gamma^v(\mathbf{u}_h^+, \nabla_h\mathbf{u}_h^+) - \delta\left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right),$$
$$\boldsymbol{\rho}_\Gamma(\mathbf{u}_h) = -\theta\left(G_\Gamma^\top(\mathbf{u}_h^+) : \left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\right)\otimes\mathbf{n}\right)^\top.$$

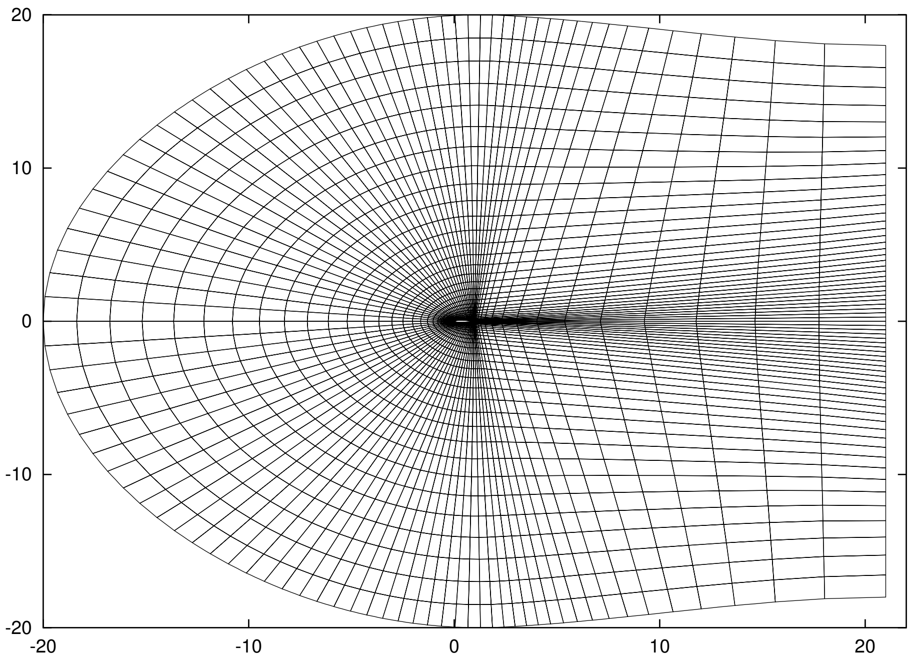
Laminar test case

$M = 0.5$, $Re = 5000$, $\alpha = 0$ flow around the NACA0012 airfoil

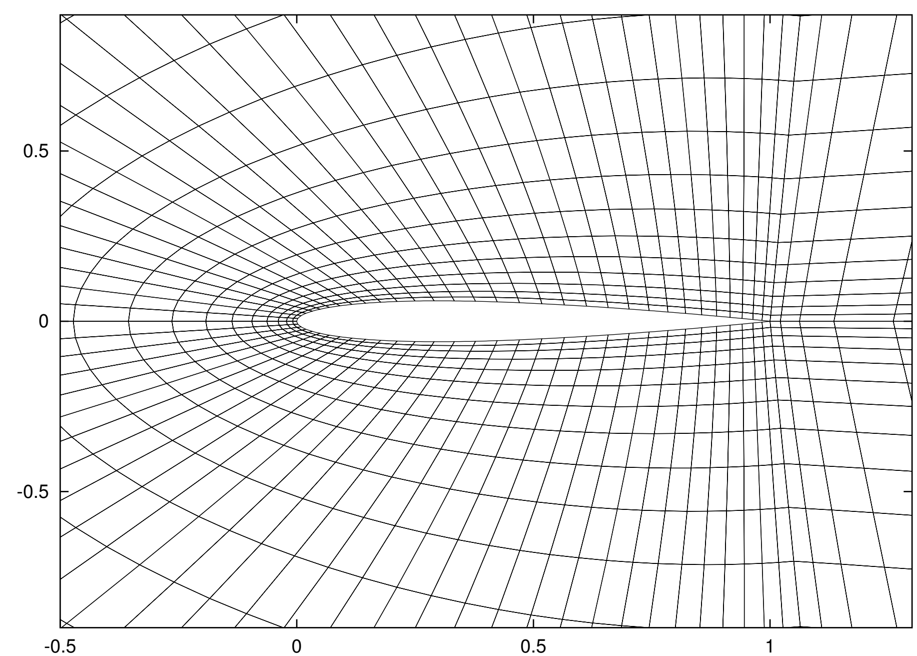
Laminar test case

$M = 0.5$, $Re = 5000$, $\alpha = 0$ **flow around the NACA0012 airfoil**

Grid of 3072 cells:



Zoom of this grid:



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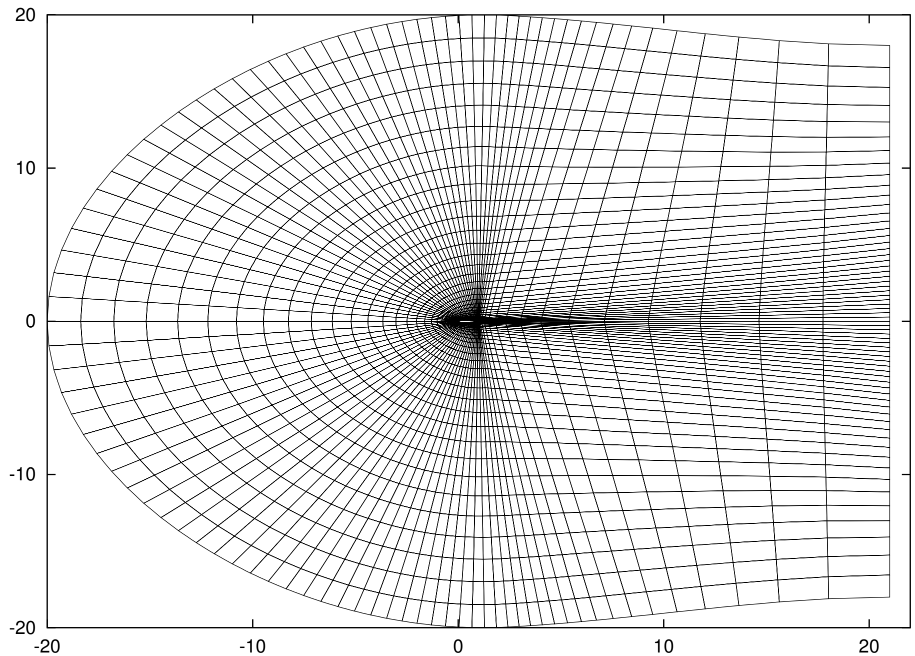

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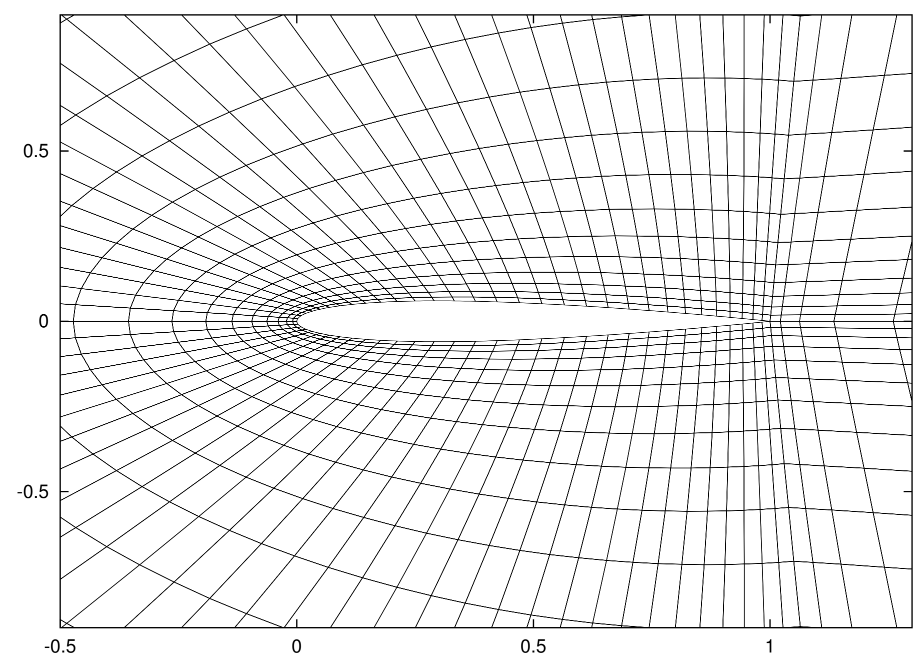
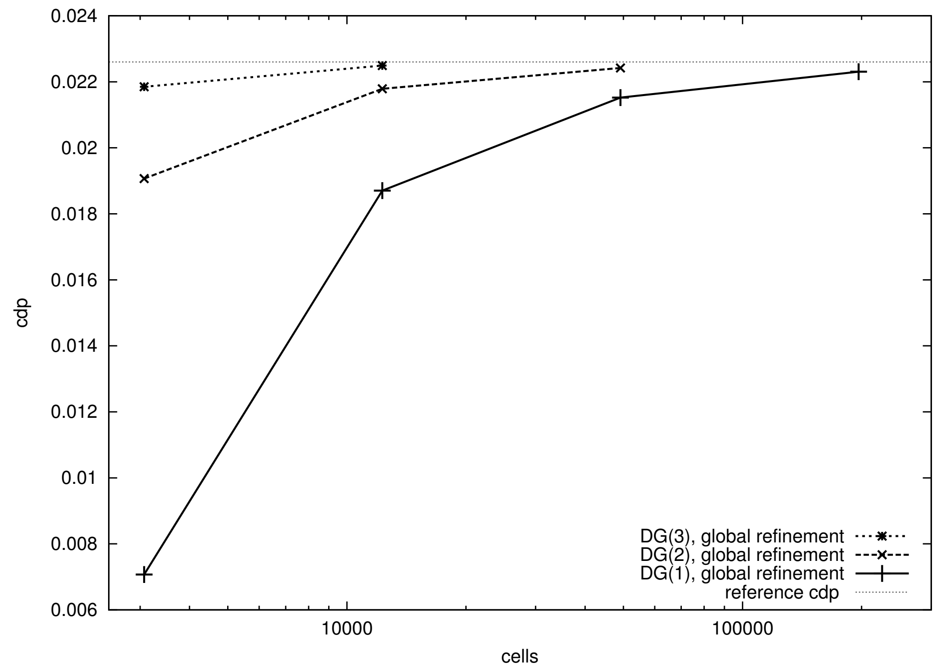


Computation using DG(p), $p = 1, 2, 3$

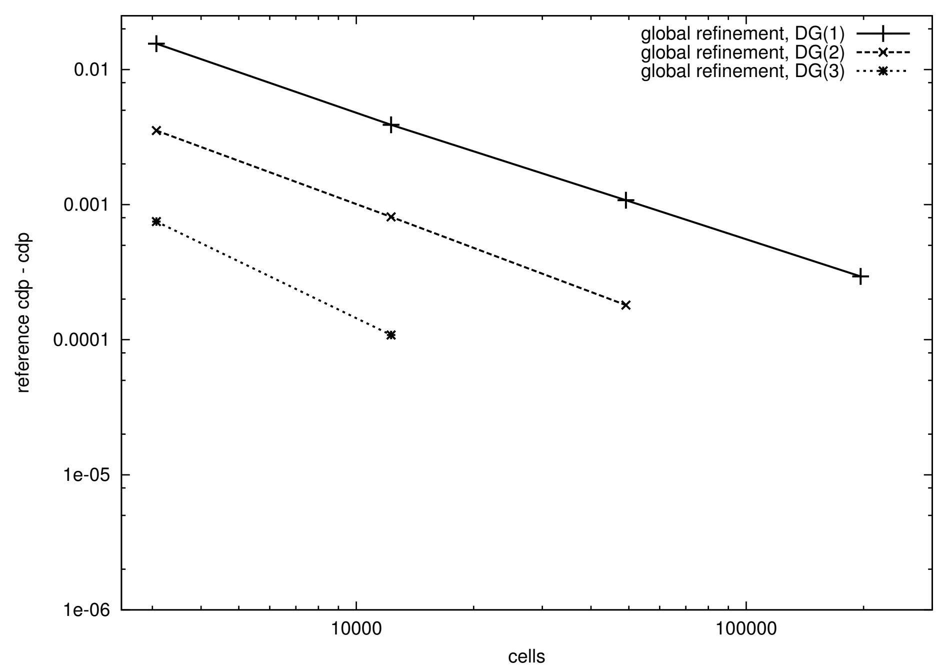
Local refinement by residual-based indicators

Test case: NACA0012, $M = 0.5$, $\alpha = 0$, $Re = 5000$

cdp



reference cdp - cdp



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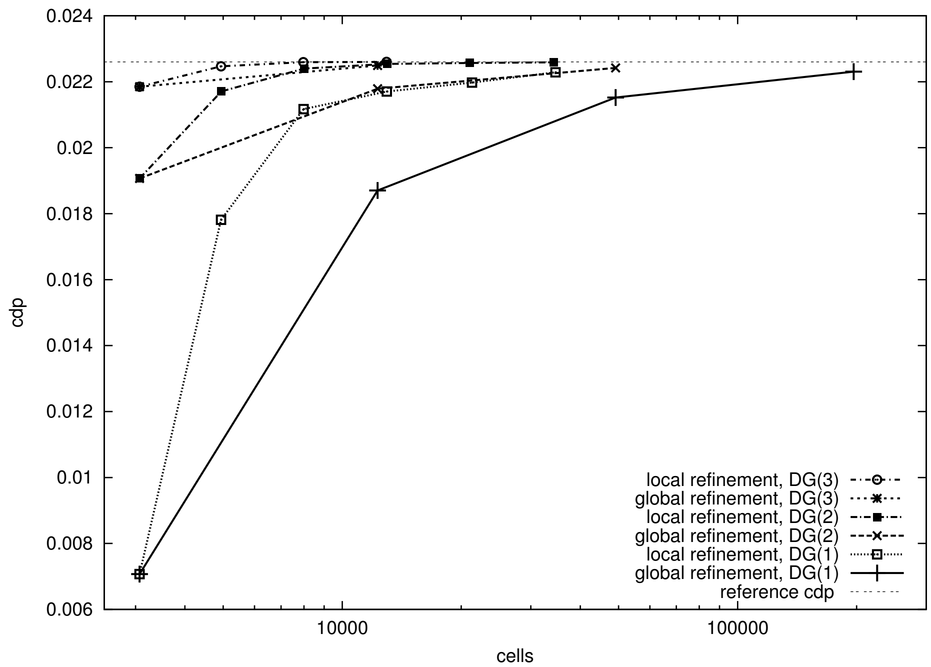

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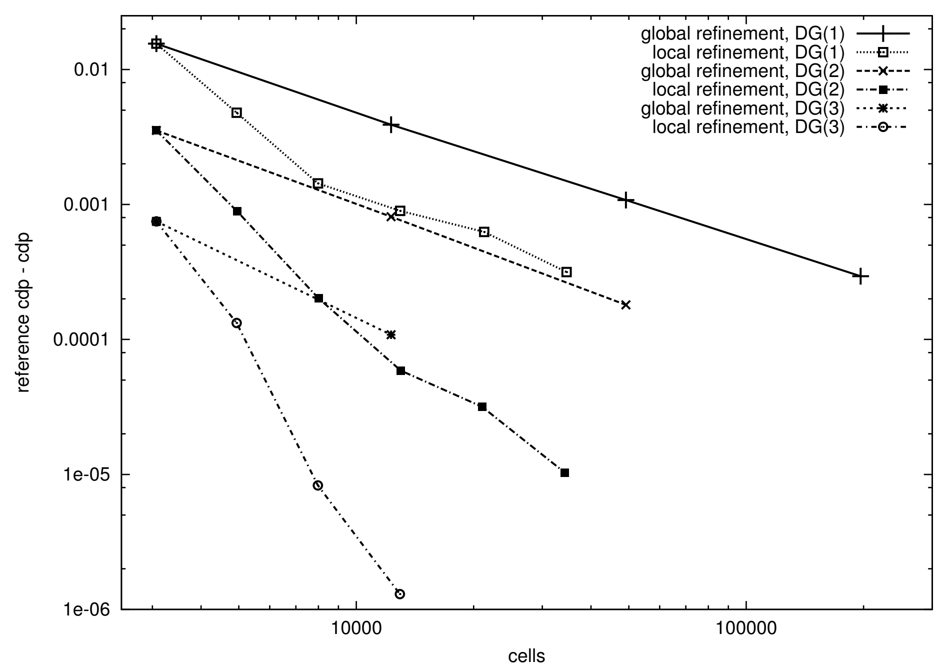

Error estimation and adjoint-based refinement

Error representation with respect to a target quantity $J(\mathbf{u})$:

$$J(\mathbf{u}) - J(\mathbf{u}_h) = -\mathcal{N}(\mathbf{u}_h, \mathbf{z}),$$

where $\mathbf{z}$ is the solution to the *adjoint* problem.

Replace $\mathbf{z}$ by the solution $\tilde{\mathbf{z}}_h$ to the discrete adjoint problem: find $\tilde{\mathbf{z}}_h \in \mathbf{V}_{h,\tilde{p}}$ such that

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}, \tilde{\mathbf{z}}_h) = J(\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{V}_{h,\tilde{p}}.$$

to obtain

$$J(\mathbf{u}) - J(\mathbf{u}_h) \approx -\mathcal{N}(\mathbf{u}_h, \tilde{\mathbf{z}}_h) =: \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa =: \eta$$

Use the *adjoint-based* indicators η_κ for adaptive mesh refinement.

Use the error estimate η to improve the computed value $J(\mathbf{u}_h)$:

$$\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_h) + \eta$$

Error estimation and goal-oriented refinement

Viscous flow at $M = 0.5$, $Re = 5000$, $\alpha = 0$ around NACA0012 airfoil

Target quantity: $J(\mathbf{u}) = c_{dp}$

# cells	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\eta = \sum_K \eta_K$	θ
3072	1.55e-02	1.06e-02	0.68
4941	4.48e-03	3.91e-03	0.87
8133	1.28e-03	1.17e-03	0.92
13521	3.33e-04	3.15e-04	0.95
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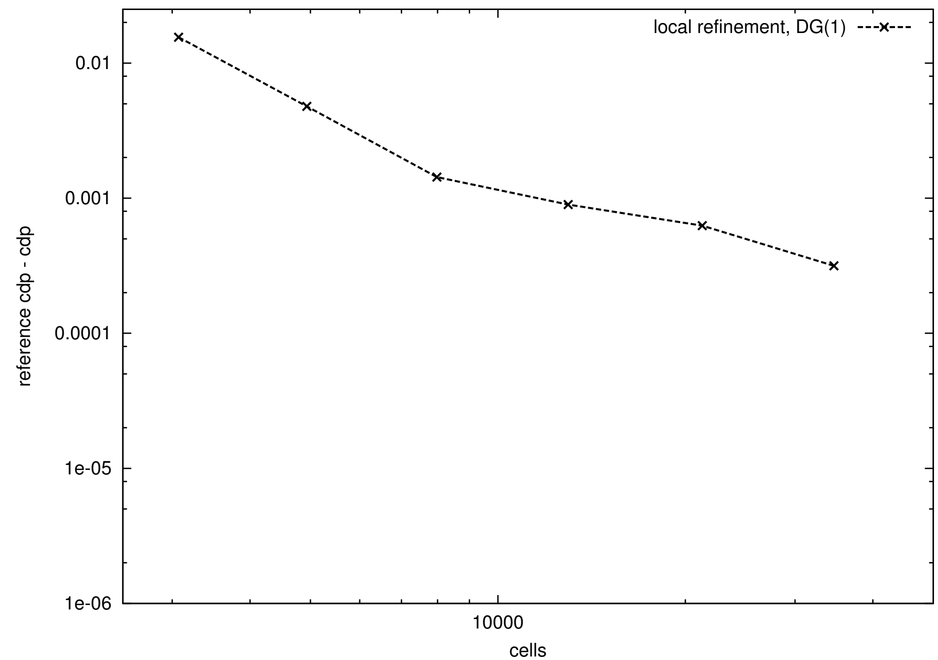

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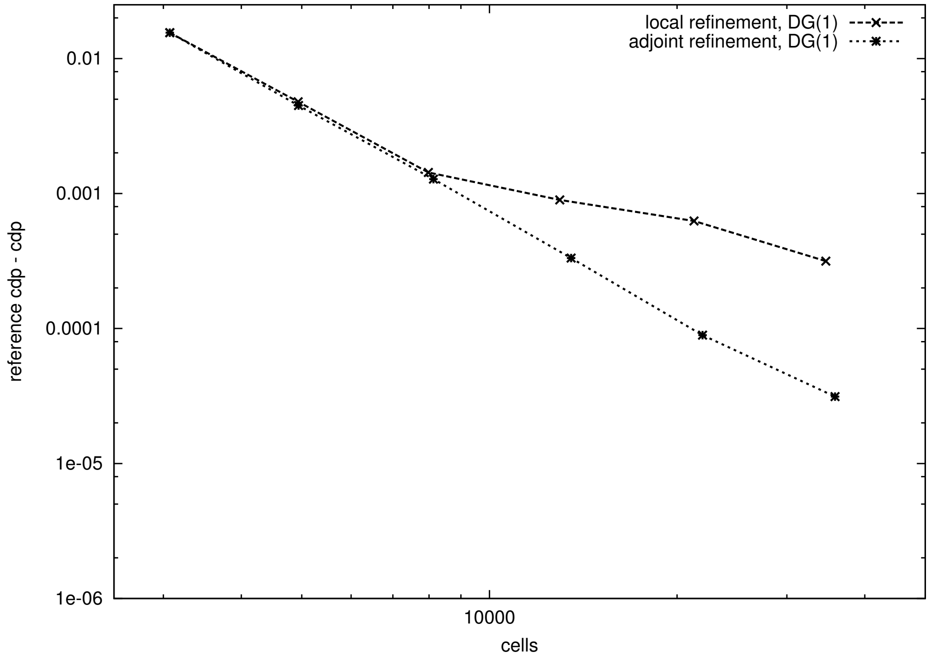

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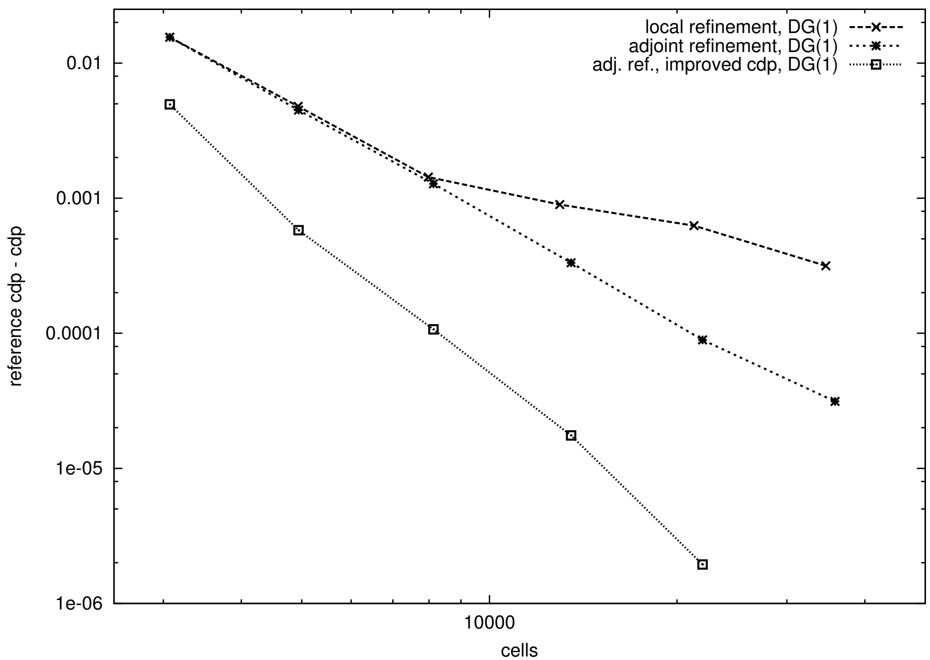


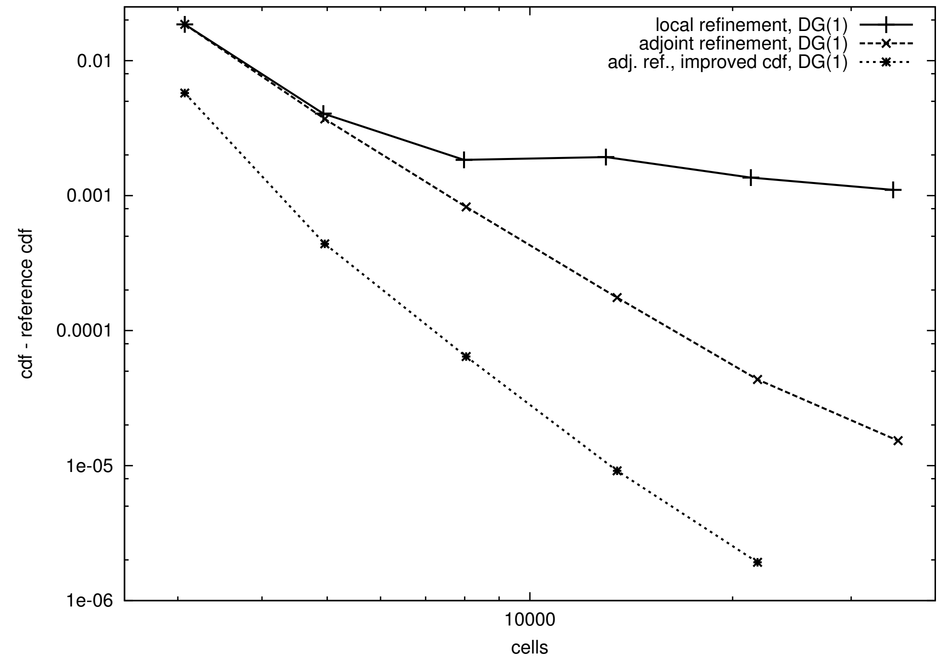
Error estimation and goal-oriented refinement

Viscous flow at $M = 0.5$, $Re = 5000$, $\alpha = 0$ around NACA0012 airfoil

Target quantity: $J(\mathbf{u}) = c_{df}$

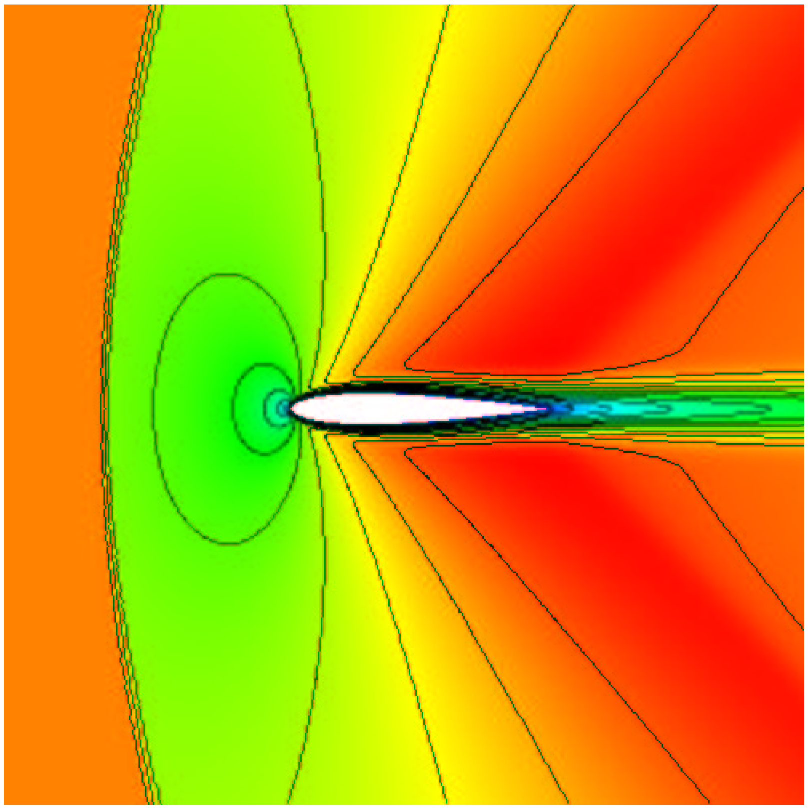
# cells	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\eta = \sum_K \eta_K$	θ
3072	-1.856e-02	-1.282e-02	0.69
4959	-3.699e-03	-3.260e-03	0.88
8040	-8.245e-04	-7.602e-04	0.92
13473	-1.756e-04	-1.665e-04	0.95
21783	-4.337e-05	-4.148e-05	0.96
35214	-1.534e-05	-1.542e-05	1.01

cdf - reference cdf

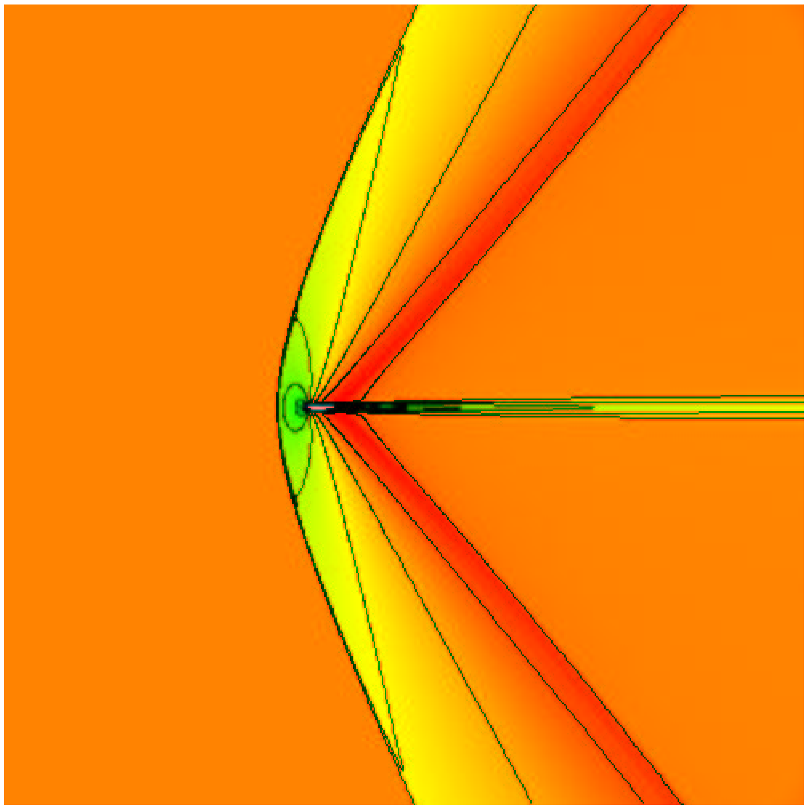


A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



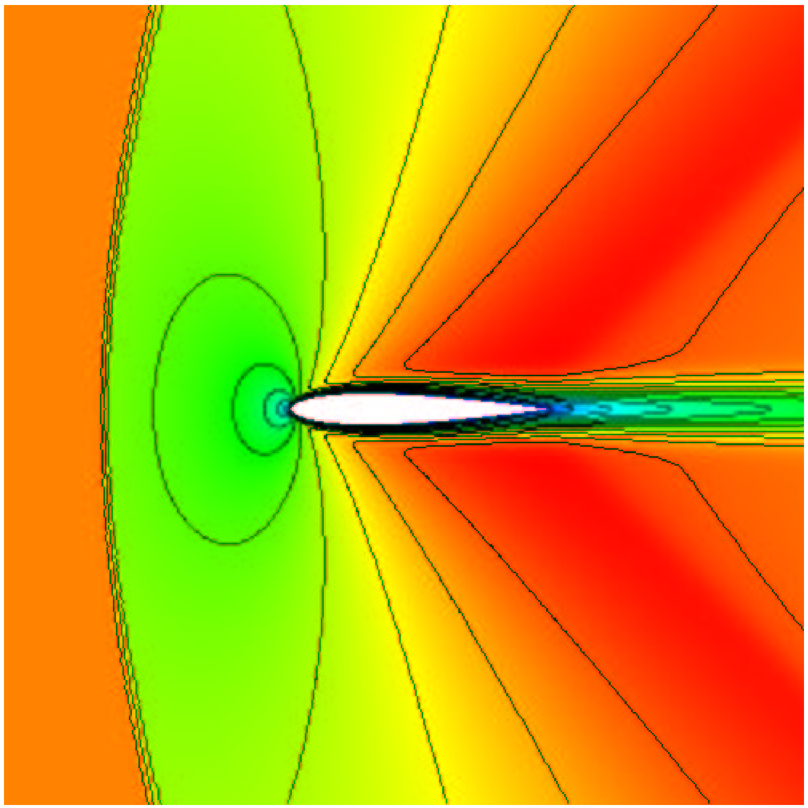
Mach isolines: close view



Mach isolines: distant view

A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil

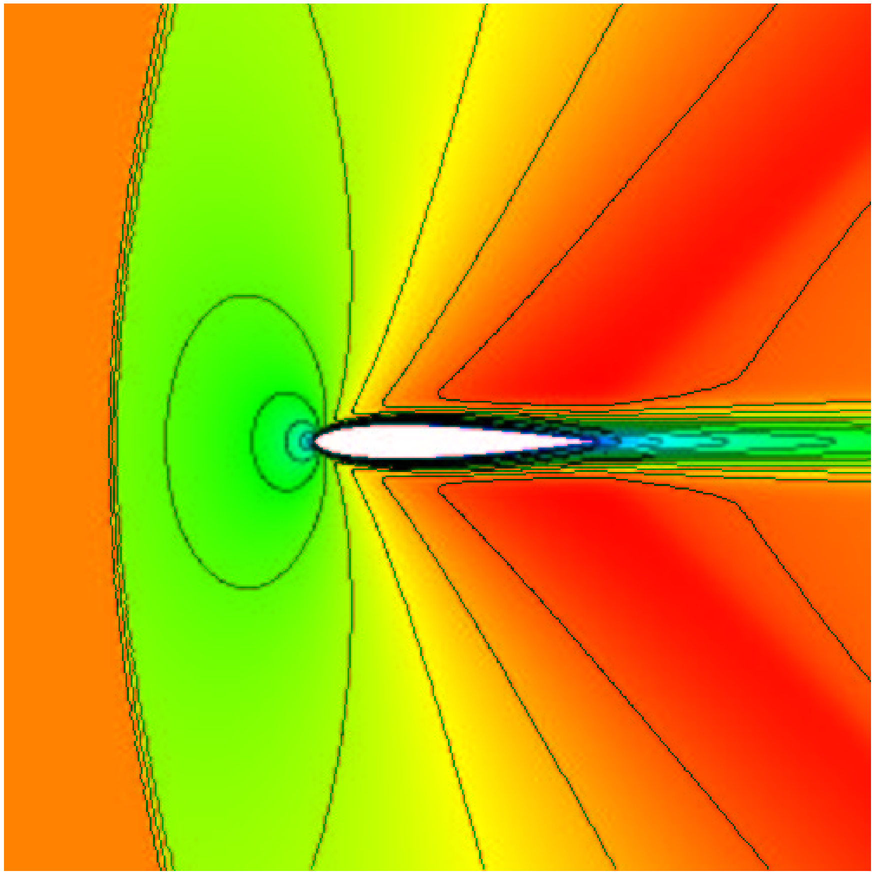


Target quantity: $J(\mathbf{u}) = c_{dp}$

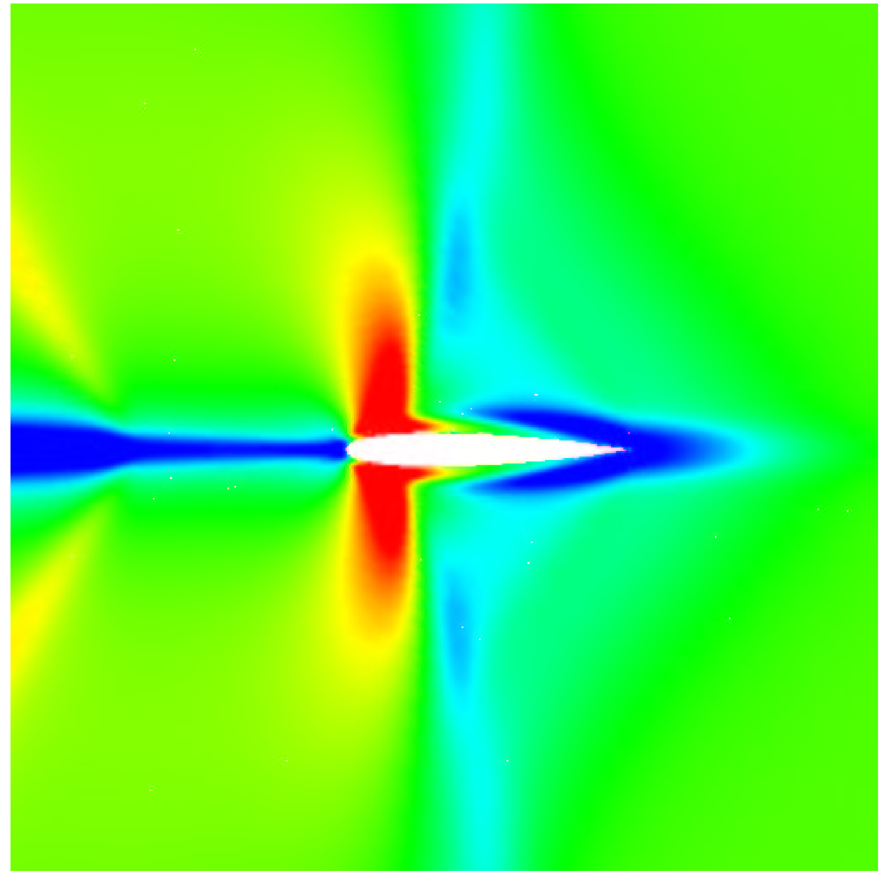
# cells	$J(\mathbf{u}) - J(\mathbf{u}_h)$	$\eta = \sum_K \eta_K$	θ
768	-1.363e-02	-6.312e-03	0.46
1260	-3.203e-03	-2.995e-03	0.94
2154	-4.844e-04	-5.368e-04	1.11
3570	-3.474e-04	-3.333e-04	0.96
6021	-1.835e-04	-1.856e-04	1.01
10038	-1.644e-04	-1.653e-04	1.01

A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



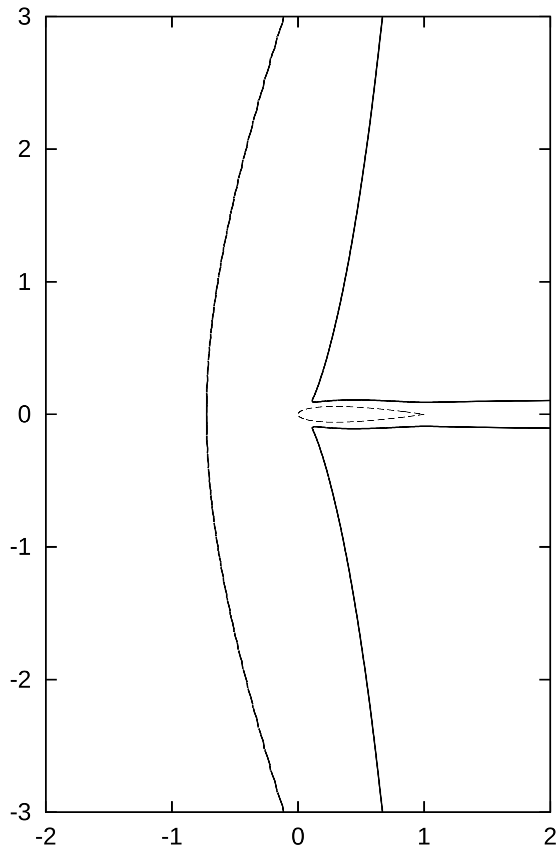
Mach isolines of primal solution



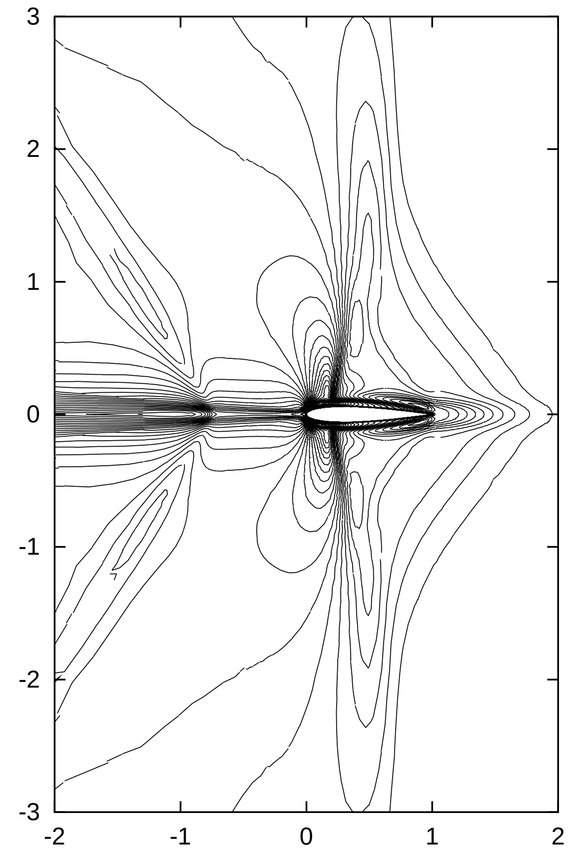
z_1 component of dual solution ($J(\mathrm{u}) = c_{dp}$)

A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



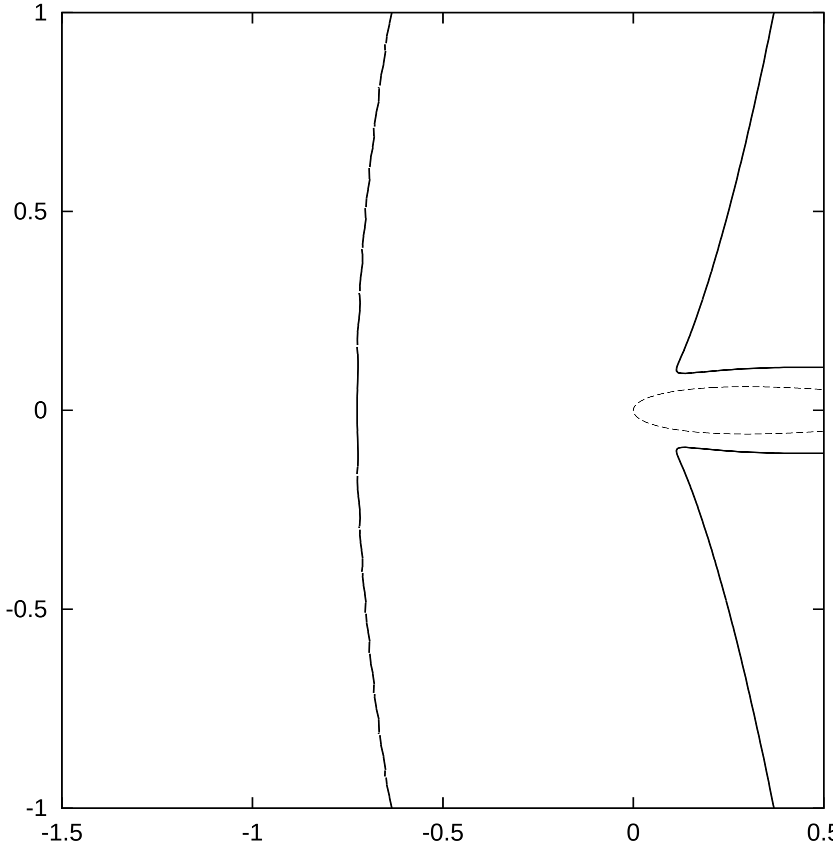
Sonic Mach isolines ($M = 1$) of primal solution



z_1 isolines of dual solution ($J(\mathrm{u}) = c_{dp}$)

A posteriori error estimation

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



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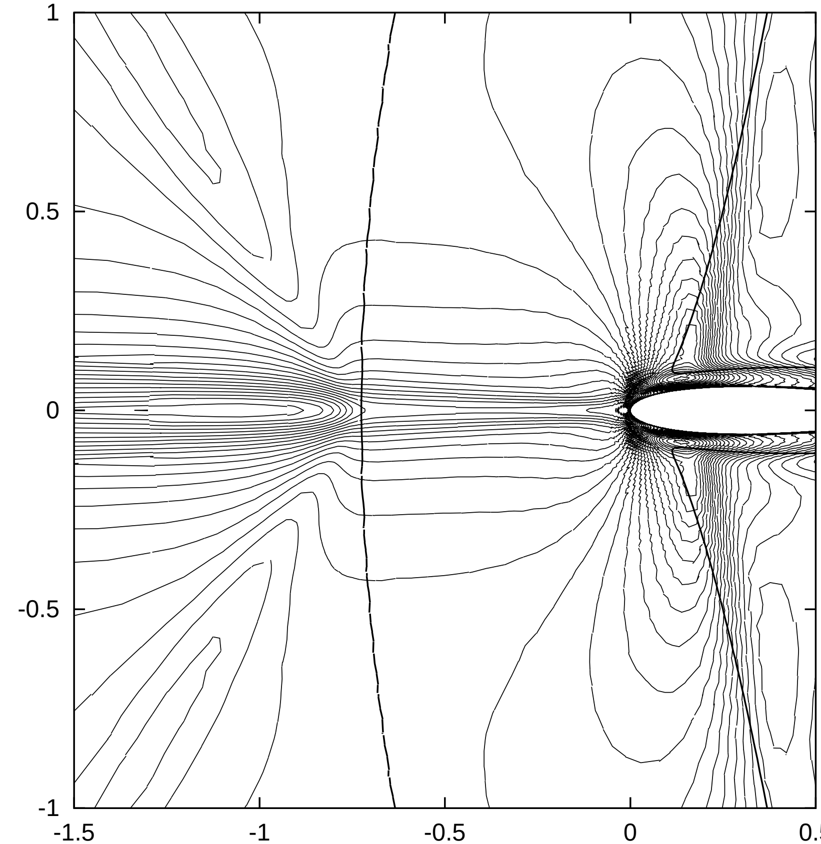
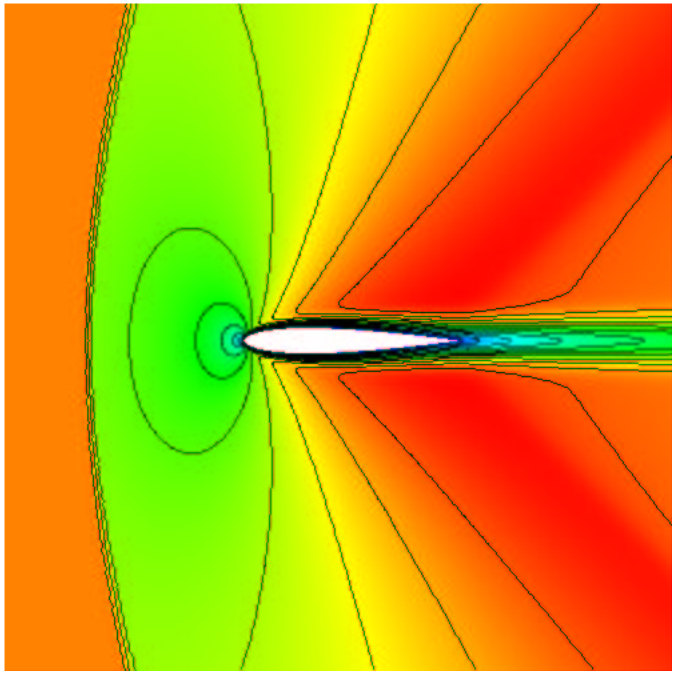


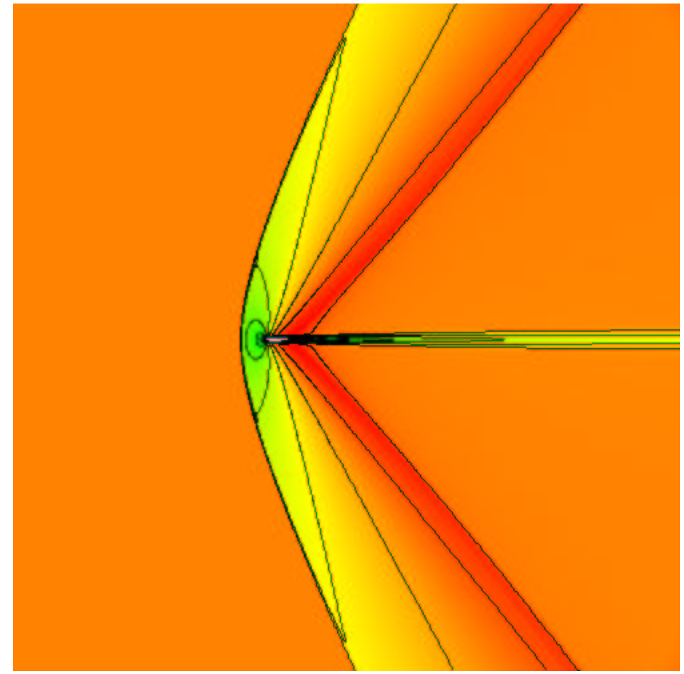
z_1 isolines of dual solution ($J(\mathrm{u}) = c_{dp}$)

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



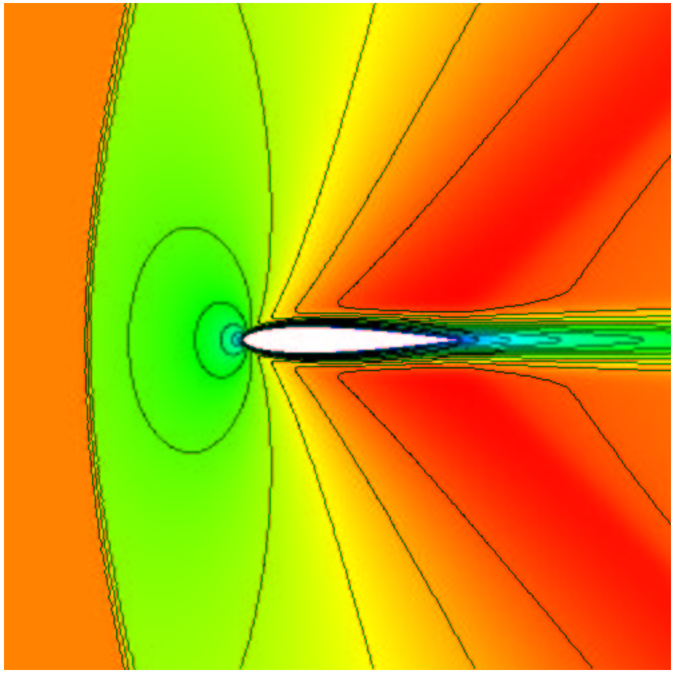
Mach isolines: close view



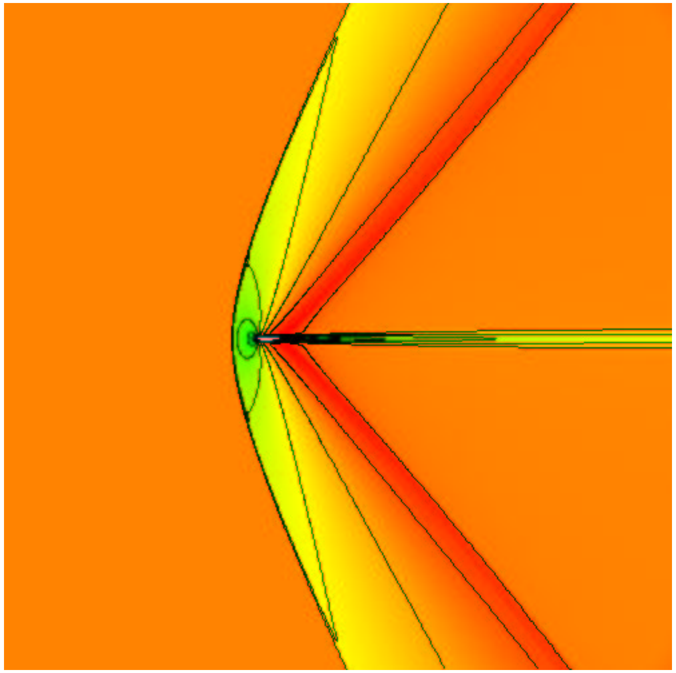
Mach isolines: distant view

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



Mach isolines: close view

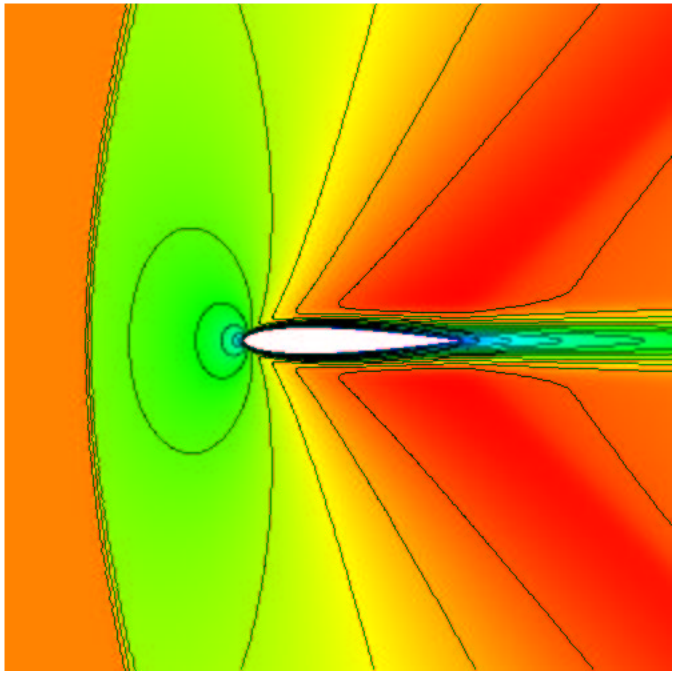


Mach isolines: distant view

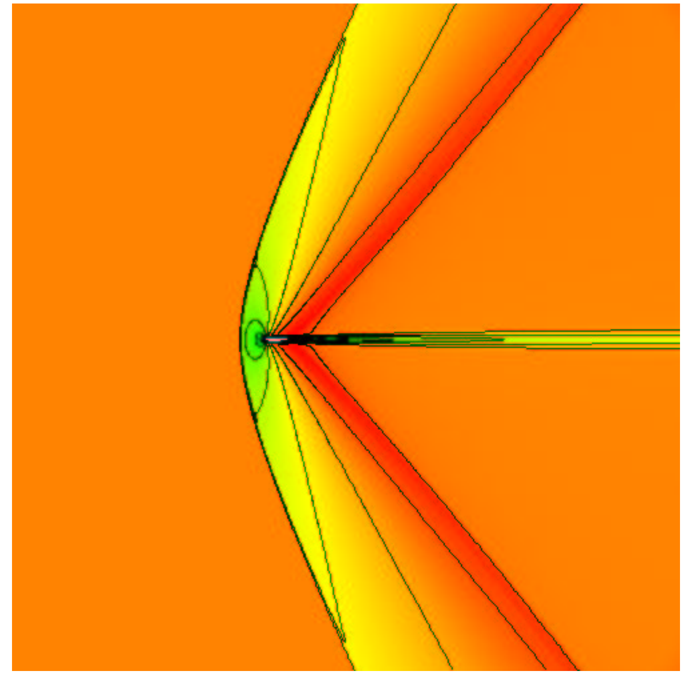
For the efficient and accurate approximation of $J(\mathrm{u}) = c_{dp}$:

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



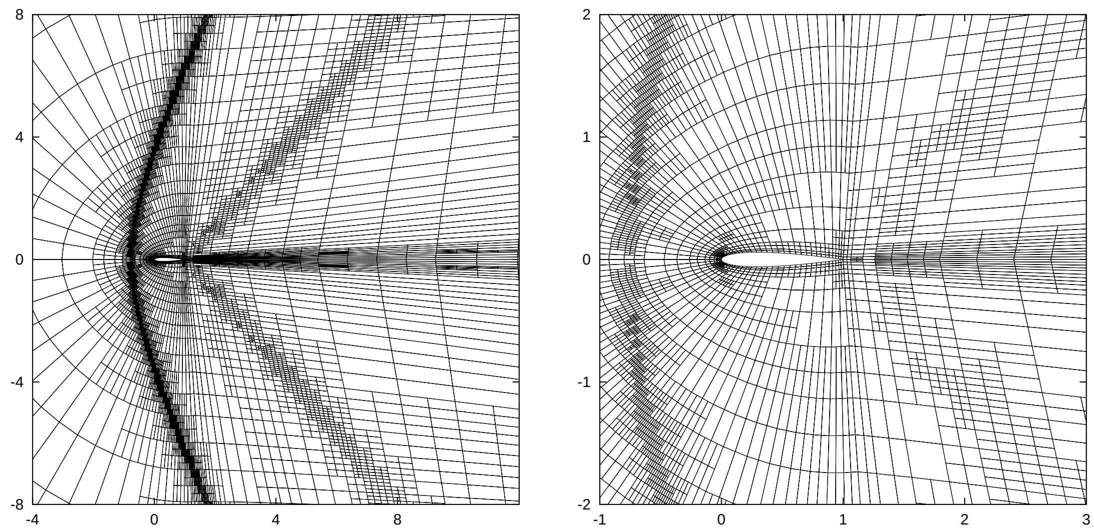
Mach isolines: close view



Mach isolines: distant view

For the efficient and accurate approximation of $J(\mathrm{u}) = c_{dp}$:

How should the mesh look like?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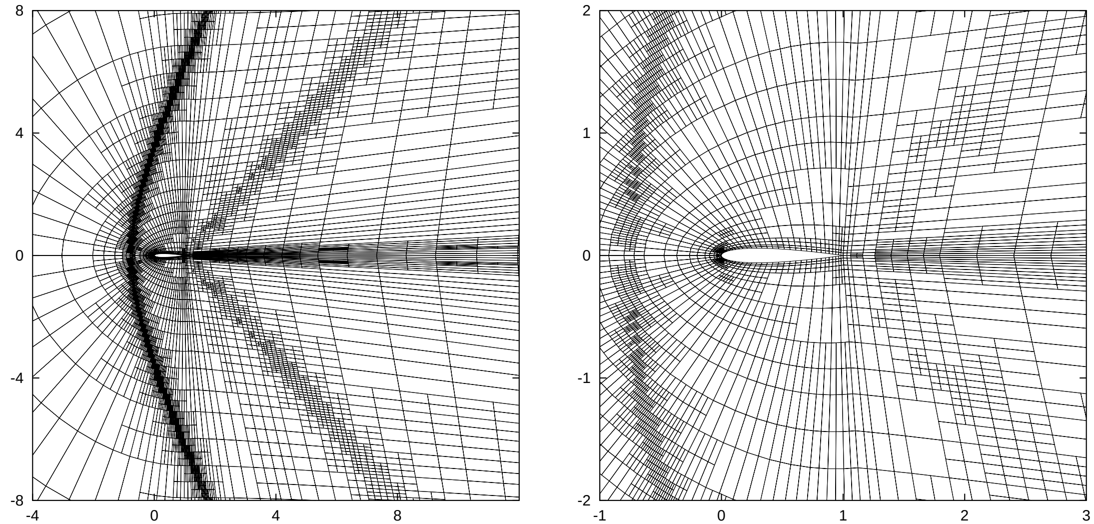
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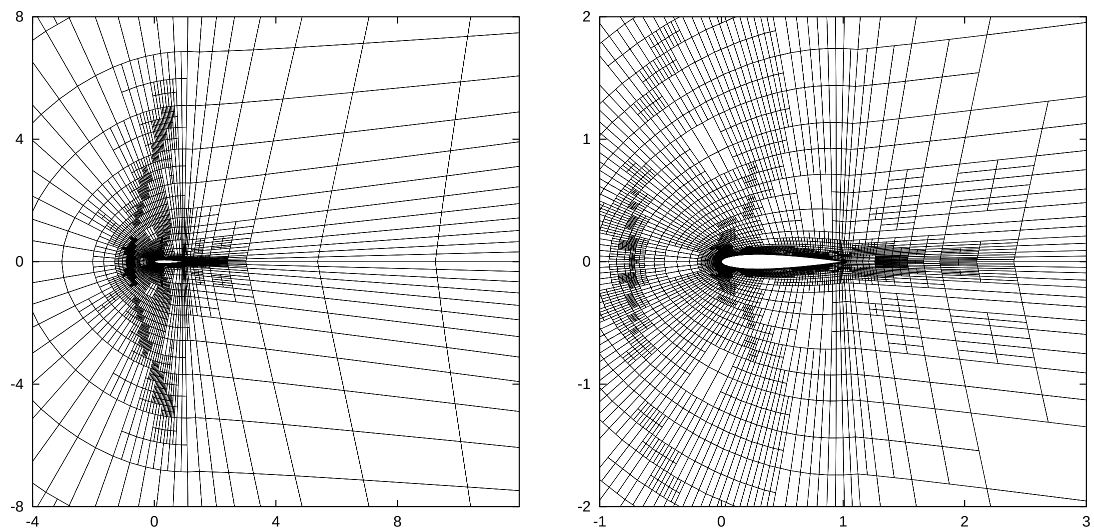
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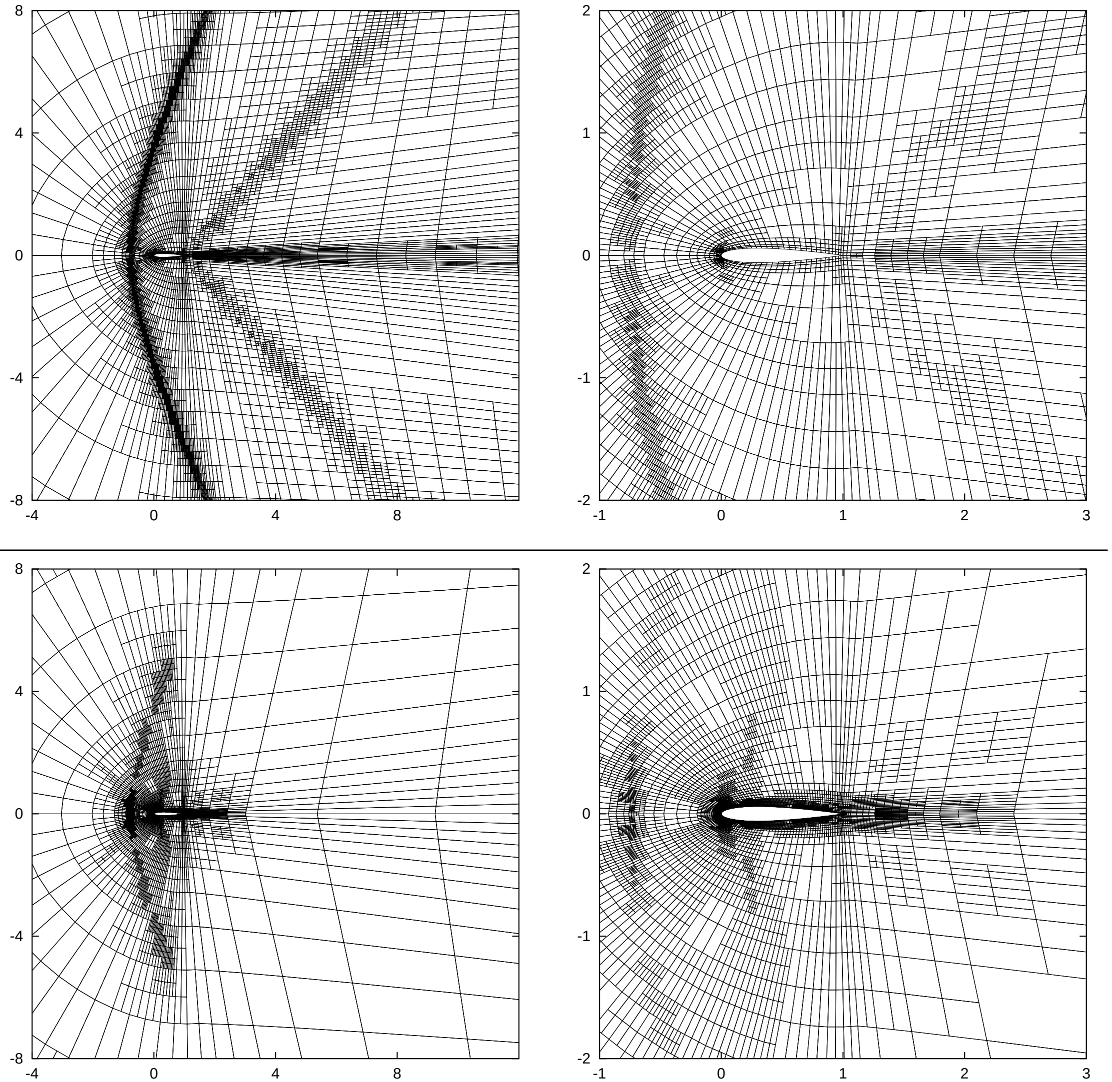
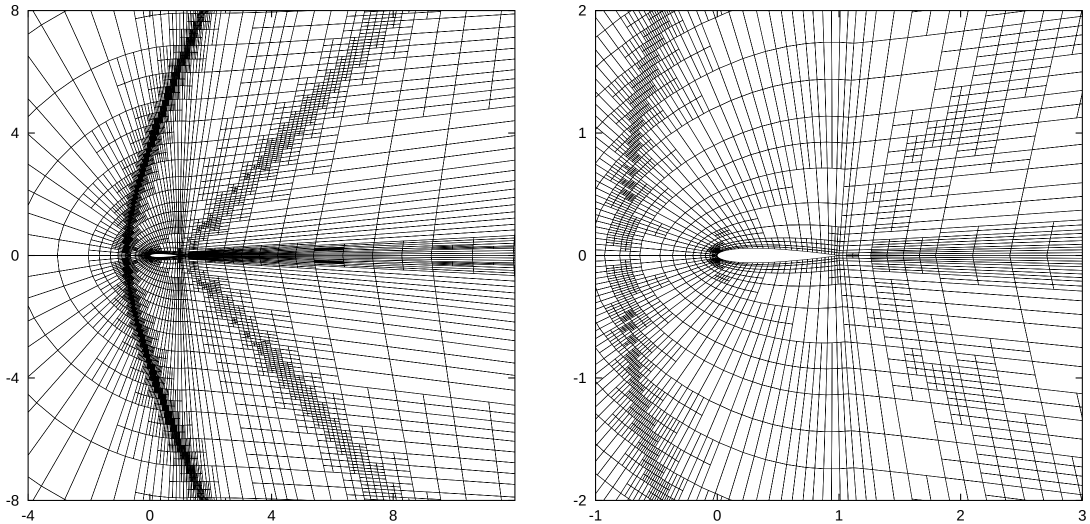


Residual-based refinement:
17670 cells with 282720 dofs
error in c_{dp} : $1.9 \cdot 10^{-3}$
error in c_{df} : $1.1 \cdot 10^{-2}$



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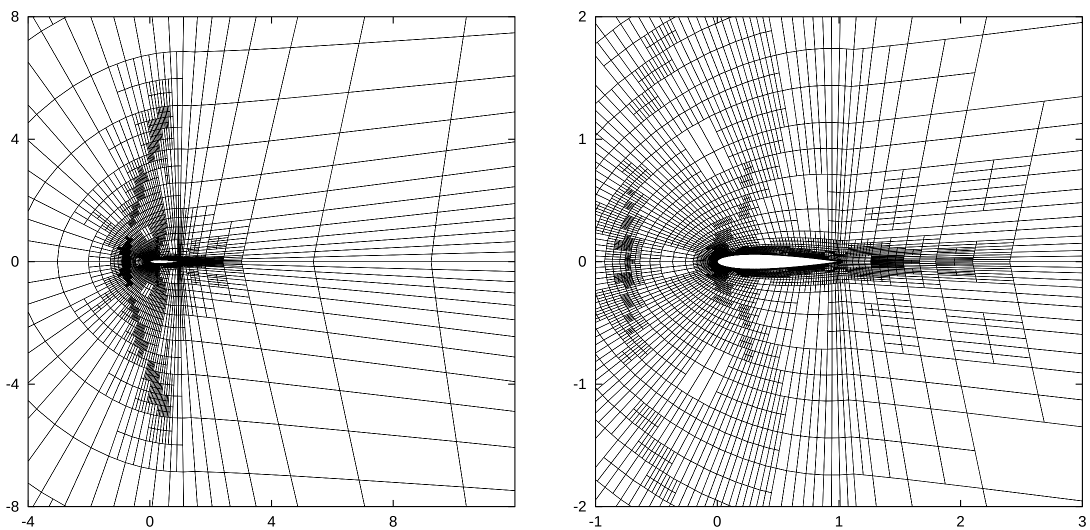


Goal-oriented refinement:
10038 cells with 160608 dofs
error in c_{dp} : $1.6 \cdot 10^{-4}$
error in c_{df} : $7.2 \cdot 10^{-4}$

Consistency and adjoint consistency

Continuous primal problem: $Nu = 0$ **in** Ω, $Bu = 0$ **on** Γ, (1)

Target functional: $J(u) = \int_\Omega j_\Omega(u)\,\mathrm{d}\mathbf{x} + \int_\Gamma j_\Gamma(u)\,\mathrm{d}s,$

with linearization: $J'[u](w) = \int_\Omega j'_\Omega[u]\,w\,\mathrm{d}\mathbf{x} + \int_\Gamma j'_\Gamma[u]\,w\,\mathrm{d}s,$

Continuous adjoint problem: $(N'[u])^* z = j'_\Omega[u]$ **in** Ω, $(B'[u])^* z = j'_\Gamma[u]$ **on** Γ, (2)

Discrete primal problem: **Find** u_h **such that** $\mathcal{N}(u_h, v) = 0 \quad \forall v \in V_h,$

Consistency: **For** u **solution to** (1) $\mathcal{N}(u, v) = 0 \quad \forall v \in V,$

Discrete adjoint problem: **Find** z_h **such that** $\mathcal{N}'[u](w, z_h) = J'[u](w) \quad \forall w \in V_h,$

Adjoint consistency: **For** z **solution to** (2) $\mathcal{N}'[u](w, z) = J'[u](w) \quad \forall w \in V.$

Consistency

Rewrite the discrete primal problem: Find u_h such that

$$\mathcal{N}(u_h, v) = 0 \quad \forall v \in V_h,$$

as follows: Find u_h such that

$$\sum_{\kappa \in \mathcal{T}_h} \int_\kappa R(u_h) v \, \mathrm{d}\mathbf{x} + \sum_{\kappa \in \mathcal{T}_h} \int_{\partial\kappa \setminus \Gamma} r(u_h) v \, \mathrm{d}s + \int_\Gamma r_\Gamma(u_h) v \, \mathrm{d}s = 0 \quad \forall v \in V_h.$$

Then, the discretization is consistent provided

$$R(u) = 0 \text{ in } \kappa, \kappa \in \mathcal{T}_h, \qquad r(u) = 0 \text{ on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \qquad r_\Gamma(u) = 0 \text{ on } \Gamma,$$

holds for the exact primal solution u.

Adjoint consistency

Rewrite the discrete adjoint problem: Find z_h such that

$$\mathcal{N}'[u](w, z_h) = J'[u](w) \quad \forall w \in V_h,$$

as follows: Find u_h such that

$$\sum_{\kappa\in\mathcal{T}_h} \int_\kappa w\, R^*(z_h)\, \mathrm{dx} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} w\, r^*(z_h)\, \mathrm{d}s + \int_\Gamma w\, r_\Gamma^*(z_h)\, \mathrm{d}s = 0 \quad \forall w \in V.$$

Then, the discretization is adjoint consistent provided

$$R^*(z) = 0 \text{ in } \kappa, \kappa \in \mathcal{T}_h, \qquad r^*(z) = 0 \text{ on } \partial\kappa \setminus \Gamma, \kappa \in \mathcal{T}_h, \qquad r_\Gamma^*(z) = 0 \text{ on } \Gamma.$$

holds for the (continuous) adjoint solution z.

Consistent modification of target functionals

Definition: $\tilde{J}(u)$ **is a consistent modification of** $J(u)$
if $\tilde{J}(u) = J(u)$ **holds for the exact solution** u**.**

We replace $J(u)$ **by** $\tilde{J}(u)$ **with**

$$\tilde{J}(u) = J(i(u)) + \int_{\Gamma} r_J(u)\,\mathrm{d}s.$$

This modification is consistent if $i(u) = u$ **and** $r_J(u) = 0$**.**

Outcome of adjoint consistency analysis

DG for the compressible Euler equations with (pressure-induced) force coefficient

$$J(\mathbf{u}_h) = \frac{1}{C_\infty} \int_{\Gamma_W} p\, \mathbf{n} \cdot \psi \, \mathrm{d}s.$$

Instead of

$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) \quad \textbf{and} \quad J(\mathbf{u}_h)$$

use

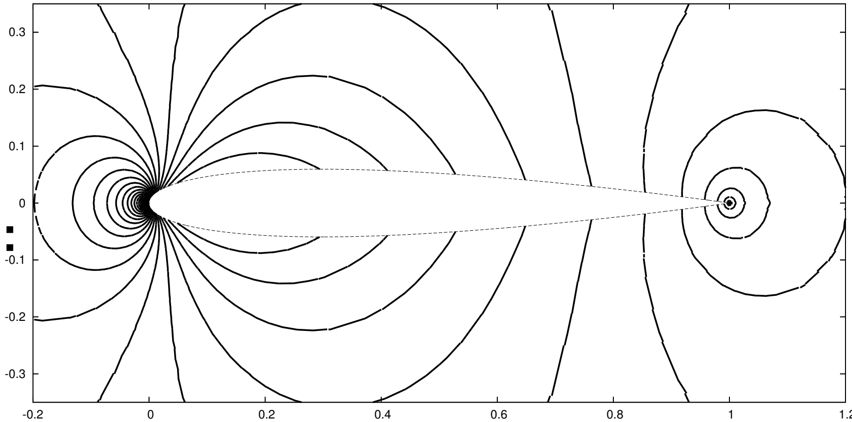
$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) \cdot \mathbf{n} \quad \textbf{and} \quad \tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h)).$$

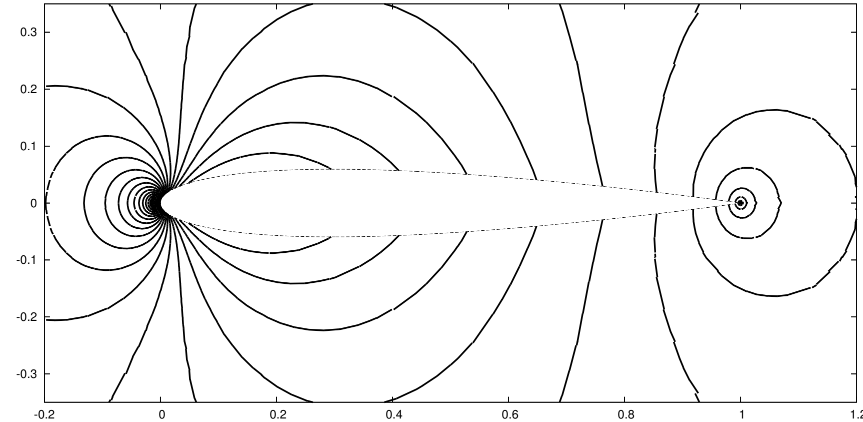
$\tilde{J}(\mathbf{u}_h)$ **is consistent as** $\mathbf{u}_\Gamma(\mathbf{u}) = \mathbf{u}$.

- **R. Hartmann. Derivation of an adjoint consistent discontinuous Galerkin discretization of the compressible Euler equations.**
  In Gert Lube and Gerd Rapin (eds.), Proceedings of the BAIL 2006 conference.
- **J. Lu and D.L. Darmofal. Dual-consistency analysis and error estimation for discontinuous Galerkin discretization: application to first-order conservation laws.**
  IMA Journal of Numerical Analysis. Submitted.

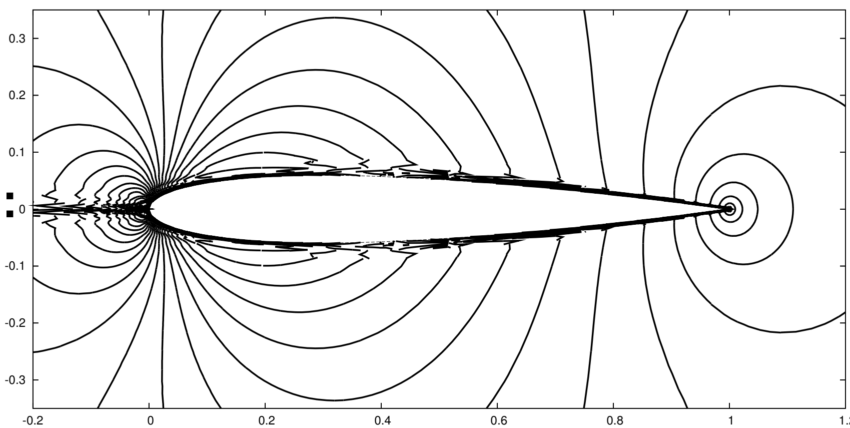
Numerical comparison: standard DG vs. adjoint consistent DG

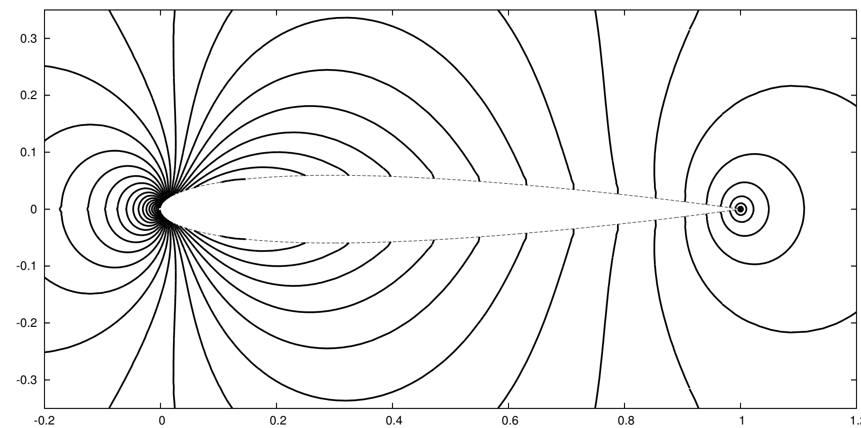
Primal solution: Mach isolines





Discrete adjoint solution: z_1 isolines

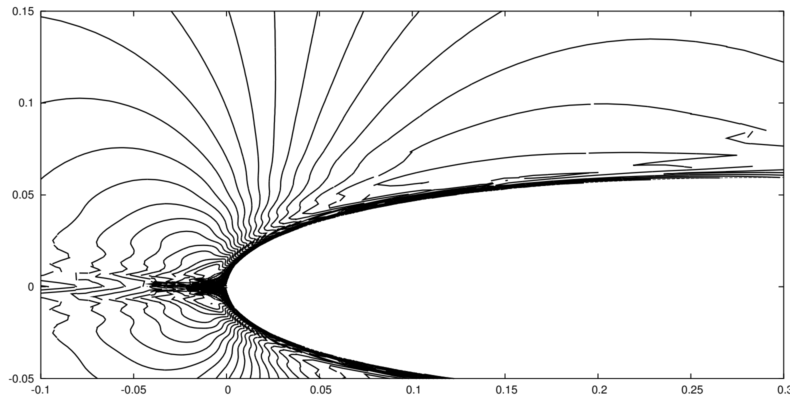




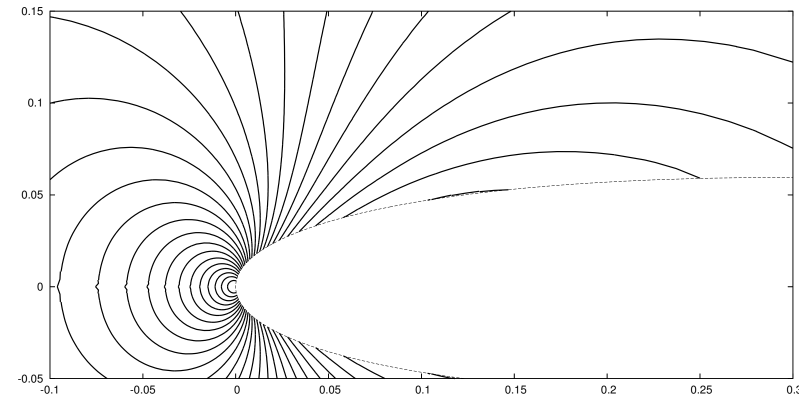
standard DG **adjoint consistent DG**

Numerical comparison: standard DG vs. adjoint consistent DG

Discrete adjoint solution: z_1 isolines



standard DG



adjoint consistent DG

primal solution: error in cdp

# elem.	standard DG	adj.cons. DG
768	-5.008e-03	-3.800e-03
1242	-1.783e-03	-8.833e-04
2061	-5.422e-04	-2.302e-04
3339	-1.617e-04	-8.405e-05
5535	-5.060e-05	-3.754e-05

Gain in accuracy: a factor of about 2

Outcome of adjoint consistency analysis

SIPG for the compressible Navier-Stokes equations with (total) force coefficient

$$J(\mathbf{u}_h) = \frac{1}{C_\infty} \int_{\Gamma_W} (p\,\mathbf{n} - \underline{\tau}\,\mathbf{n}) \cdot \psi \, \mathrm{d}s.$$

Instead of

$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}), \quad G_\Gamma(\mathbf{u}_h^+) = G(\mathbf{u}_h^+), \quad \textbf{and} \quad J(\mathbf{u}_h),$$

use

$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)) \cdot \mathbf{n}, \quad G_\Gamma(\mathbf{u}_h^+) = G(\mathbf{u}_\Gamma(\mathbf{u}_h^+)), \quad \textbf{and} \quad \tilde{J}(\mathbf{u}_h),$$

with

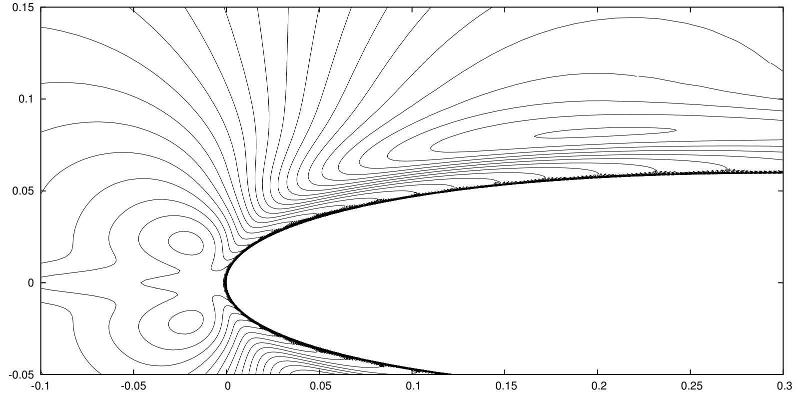
$$\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h)) + \int_{\Gamma_W} \delta \left(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+) \right) \cdot \mathbf{z}_\Gamma \, \mathrm{d}s,$$

and $\mathbf{z}_\Gamma = \frac{1}{C_\infty}(0, \psi_1, \psi_2, 0)^\top$

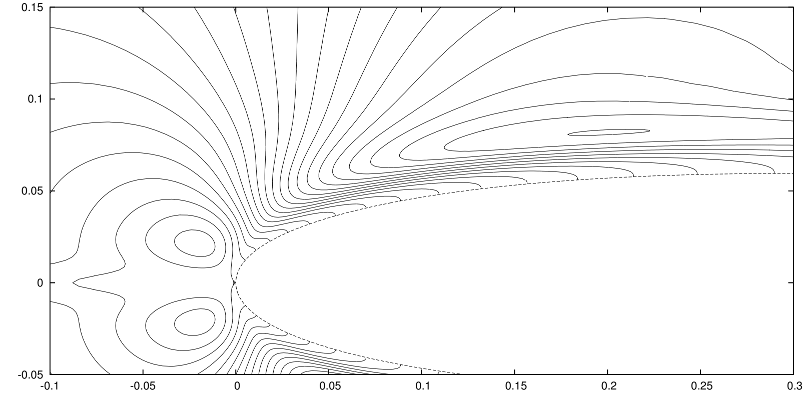
- **R. Hartmann. Adjoint consistency analysis of Discontinuous Galerkin discretizations. SIAM J. Numer. Anal., 2006. Submitted.**

Numerical comparison: standard DG vs. adjoint consistent DG

Discrete adjoint solution: z_4 isolines



standard DG



adjoint consistent DG

primal solution: error in cd

# elem.	standard DG		adj.cons. DG		ac. DG/IP mod.	
3072	-3.164e-03	-	1.502e-03	-	-1.243e-03	-
12288	8.048e-04	3.9	3.682e-05	40.8	6.994e-04	1.8
49152	4.519e-04	1.8	-1.139e-06	32.3	4.795e-04	1.5

Gain in accuracy: a factor of about 2-400!!

Conclusion

DG discretization of the compressible Euler and Navier-Stokes equations

- **Accurate error estimation with respect to aerodynamical force coefficients**
- **Comparison of residual-based and adjoint-based mesh adaptation**
- **Improved order of convergence for an adjoint consistent SIPG discretization (including a consistent modification of the target quantity)**

Outlook: Next steps towards application in industry

- Extension of the flow solver
  - to three-dimensional turbulent, high Reynolds flow
- The above extension also for
  - the computation of the adjoint solution
  - the evaluation of refinement indicators (residual-based and adjoint-based)
  - and the error estimation with respect to aerodynamical force coefficients
- Error estimation and adaptivity with respect to multiple target quantities

EU Project: ADIGMA