Construction of very high order Residual Distribution Schemes for steady problems

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Overview

1. RDS schemes
2. Construction on scalar advection
3. Numerical examples
   - Scalar
   - Systems
Forewords

• This lecture focusses on steady problems.
• Several JCP papers on second order unsteady.
• Existing work on coupling hyperbolic/convective terms to viscous one (Villedieu–Ricchiuto)

Discussion on scalar problems.
\[ \tilde{\lambda} \cdot \nabla u = 0 \quad x \in \Omega \]
\[ u = g \quad x \in \Gamma^- \]

- unstructured triangular meshes, triangles \( T \),
- \( u_\sigma \simeq u(M_\sigma), \sigma \text{ d.o.f} \)
- scheme \( u_{\sigma}^{n+1} = u_{\sigma}^n - \omega_i \sum_{T, \sigma \in T} \Phi_T^\sigma. \)
- steady solution i.e. \( n \to +\infty \).
Second order

- \( \sigma \): vertices of the triangles
- \( u^h \): piecewise linear interpolant of \( \{ u_\sigma \} \),
- Conservation relation

\[
\sum_{\sigma \in T} \Phi_{\sigma} = \int_T \vec{\lambda} \cdot \nabla u^h dx := \Phi^T
\]

\[
= \sum_{\sigma \in T} k^T_{\sigma} u_{\sigma} \quad k_{\sigma} = \int_T \vec{\lambda} \cdot \nabla N_{\sigma}
\]
Example: piecewise interpolation

- Upwind finite volume scheme,
- N scheme

\[ \Phi_\sigma = k_\sigma^+ (u_\sigma - \tilde{u}) \]
\[ \tilde{u} = \left( \sum_{\sigma'} k_{\sigma'}^- \right)^{-1} \left( \sum_{\sigma' \in T} k_{\sigma'}^- u_{\sigma'} \right) \]

- Local Rusanov,

\[ \Phi_\sigma = \frac{1}{3} \left( \Phi^T + \alpha \sum_{\sigma' \neq \sigma, \sigma \in T} (u_\sigma - u_{\sigma'}) \right) \]
Example: piecewise interpolation

The upwind FV scheme, N scheme and Rusanov scheme are monotone schemes

\[ \Phi_\sigma = \sum_{\sigma' \neq \sigma} c_{\sigma\sigma'}(u_\sigma - u_{\sigma'}), \quad c_{\sigma\sigma'} \geq 0. \]

Provides a local maximum principle on the solution
Summary

• Conservation property

\[
\sum_{\sigma \in T} \Phi_{\sigma} = \int_T \vec{\lambda} \cdot \nabla u^h \, dx := \Phi^T
\]

• Second order approximation of the flux (here because \(u^h = u + O(h^2)\)).

• LP condition \((\Phi_T^\sigma(u) = O(h^3)\) for the exact solution\)

• Monotonicity preserving scheme \((\Phi_{\sigma} = \sum_{\sigma'} c_{\sigma\sigma'} (u_{\sigma} - u_{\sigma'})\)

• technique for going from first order to second order.

scheme: \[
\sum_{T, \sigma \in T} \Phi_T^\sigma = 0
\]
How to get really high order, compact, monotonicity preserving schemes on general meshes?

- Use of $P^k$ interpolation,
- “Improved” conservation relation
- Probably more general than this (Hermite, spectral, etc)
Other contributions

• Abgrall-Roe, J. Scientific Computing, 2001
• Hubbard (Computer and Fluids, 2005)
• Ricchiuto et al (VKI LS, 2005)
• Andrianov et al. (IJNM, 2005)
• De Palma et al. (JCP in press)
• …
Notations–degrees of freedom

- mesh $\tau$, triangles $T$, vertices $M_j$,
- we seek a solution that is piecewise polynomial of degree $k$ in each triangle,
- need to provide $(k + 1)(k + 2)/2$ degrees of freedom:

Scheme

$$\sum_{\sigma \in T} \Phi^T_\sigma = 0$$
Example, $k = 2$

Degree of freedom $P^2$ interpolation.
Structural condition:

Lax Wendroff like result + high order

- $\Phi^T := \int_T \text{div} \, F^h(u^h) \, dx = \int_T \vec{\lambda} \cdot \nabla u^h \, dx = \sum_{\sigma \in T} \Phi^T_\sigma(u^h)$,

  + standard assumption Lax Wendroff solution of

  \[
  \text{div} \, F(u) = 0 \quad + \text{boundary conditions}
  \]

- High order if $\Phi^T_\sigma(u^h) = O(h^{2+k})$ for smooth solutions + regular meshes
Monotonicity condition

Ensures stability in $L^\infty$

$$\Phi_\sigma = \sum_{\sigma' \neq \sigma, \sigma' \in T} c_{\sigma \sigma'} (u_\sigma - u_{\sigma'})$$

with

$$c_{\sigma \sigma'} \geq 0$$

dependant on the solution.
Previous try (Abgrall-Roe, JSC, 2001)

- define sub-triangulation
- use a “reference” monotone first order scheme $\Phi_{L,T'}^{L,T'}$ in each sub–triangulation
- define sub–residuals in each sub-triangle accordingly

\[
 u_{\sigma}^{n+1} = u_{\sigma}^{n} - \omega_i \sum_{T,\sigma \in T} \Psi_{\sigma}^{T}, \quad \Psi_{\sigma}^{T} = \sum_{T' \subset T, \sigma \in T'_T} \left( \Phi_{\sigma}^{T'} \right)^* 
\]
First try

- We can construct the $N$ scheme on the sub–triangles of $T$: $\Phi_{\sigma}^{T_\xi}$ for $\xi = I, II, III, IV$

- Consider
  \[ \Phi_\xi = \int_{T_\xi} \vec{\lambda} \cdot \nabla u^h. \]  
  quadratic

- clear that linear $\Phi_\xi \neq \sum_{\sigma \in T_\xi} \Phi_{\sigma}^{T_\xi}$

- but we still can use the N scheme for a comparison purpose and want .

  \[ \Psi_{\sigma}^{T_\xi} = \beta_\xi^\sigma \Phi_\xi \]
Main problem:

\[ \sum_{\sigma \in T'} \Phi^{L,T}_{\sigma} \neq \int_{T'} \vec{\lambda} \cdot \nabla u^h = \sum_{\sigma' \in T'} \left( \Phi^{T'}_{\sigma T} \right)^* \]

Problem is not conservation but algebraic
Construction of $\beta^{T'}_{\sigma}$ needs this condition.
An always defined solution

- Start (for example) from the Rusanov scheme
- “limit” residuals
- update
Rusanov scheme

\[
\Phi^T_\sigma = \frac{1}{6} \left( \int_{\partial T} [\vec{\lambda} \cdot \vec{n}] u^h \, dx + \alpha_T \sum_{\sigma' \in T} (u_{\sigma'} - u_\sigma) \right)
\]

with

\[
\alpha_T \geq \max_{\sigma \in T} \left| \int_T \vec{\lambda} \cdot \nabla N_\sigma \, dx \right|.
\]

monotone scheme.
For Rusanov scheme,

\[
\sum_{\sigma \in T} \Phi_T^\sigma = \int_{\partial T} [\vec{\lambda} \cdot \vec{n}] u^h dx = \Phi_T.
\]

• define \( x_\sigma = \Phi_R^\sigma / \Phi_T \),

• set \( \beta_T^\sigma = \frac{x_\sigma^+}{\sum_{\sigma'} x_{\sigma'}^+} \) and \( \Phi_T^\sigma := \beta_T^\sigma \Phi_T \).

always defined because \( \sum_{\sigma'} \Phi_{H,T}^\sigma = \Phi_T \) implies \( \sum_{\sigma'} x_{\sigma'} = 1 \) so that \( \sum_{\sigma'} x_{\sigma'}^+ \geq 1 \).
Properties

If there exists a unique solution to

\[
\sum_{T \ni \sigma} \Phi_{\sigma}^{H,T} = 0 \quad \sigma \in \Omega \quad (\vec{\lambda} \cdot \nabla u = 0)
\]

\[u_{\sigma} = g\] on \(\Gamma^{-}\)

then the scheme is third order accurate because

\[
\int_{\partial \Gamma \setminus \Gamma^{-}} \vec{\lambda} \cdot \nabla u^h \, dl = O(h^{3+1})
\]
Numerical experiments

convection
Numerical experiments

convection

solid rotation
Same second order

convection

solid rotation
Why: existence of mild spurious modes

- Interpretation
Interpretation

First order scheme

\[ \tilde{\lambda} \]
This problem does not exist for genuinely upwind schemes
Fix

Force some upwinding: add

\[ \Theta(u^h)h \int_T \left( \bar{\lambda} \cdot \nabla N_\sigma \right) \left( \bar{\lambda} \cdot \nabla u^h \right) \]
Scheme

\[
\sum_{T \ni \sigma} \left( \Phi_{\sigma}^{H,T} \right)^* = 0 \quad \sigma \in \Omega \quad (\vec{\lambda} \cdot \nabla u = 0)
\]

\[
u_{\sigma} = g \quad \text{on } \Gamma^-
\]

with

\[
\left( \Phi_{\sigma}^{H,T} \right)^* = \beta_{\sigma}^T \left( \int_T \vec{\lambda} \cdot \nabla u \, dx \right) + \Theta(u^h)h \int_T \left( \vec{\lambda} \cdot \nabla N_{\sigma} \right) \left( \vec{\lambda} \cdot \nabla u^h \right)
\]

Still third order accurate, not any more (formally) monotonicity preserving

but essentially non oscillatory
Numerical experiments

convection

solid rotation
Fourth order

Same. Replace quadratic element by cubic elements (10 dof/element)
Accuracy : convection

Grid Convergence
Accuracy/ # of dof
Burgers equation (4th order)

\[ \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = 0, \quad u(x, y) = 1.5 - 2x \text{ on inflow boundaries} \]
Burgers equation (2nd/3rd order)

same number of dof.
Comparison, 1

- Fourth order
- Second order
Comparison, 2

2nd order 4th order
Fluid Mechanics examples


• Additionnal stabilisation: formal extension to systems
Jet (3rd order)
Jet (3rd order) without dissipation
O1/O2/O3, same dof
4 state shock tube
4 state shock tube

101 × 101

201 × 201
Conclusions, perspectives.

- Residual distribution of high order (3rd, 4th),
- Essentially non oscillatory,
- Preliminary results for fluid mechanics
To be done

• Avoid additional stabilisation ? (system)
• Boundary treatement
• Efficiency
• Unsteady
To be done

- Avoid additional stabilisation (system)
- Boundary treatment
- Efficiency
- Unsteady

ADIGMA!