Analysis of a hybrid p-Multigrid method for the discontinuous Galerkin discretisation of the Euler equations

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Outline

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- Solvers
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  - h- or p-Multigrid
    - General FAS framework
    - Theoretical analysis
- Comparison: flow around NACA0012
- Conclusions And Future Work

Motivation

Advantages of higher order DGFEM

- increased order improves ratio precision wrt RAM and CPU
- maintains first order stencil on “unstructured level”
  - parallellisability
  - low overhead associated to mesh
- data per element is locally “structured”
  - no global renumbering of unknowns
  - casting of global operator into matrix-matrix operations
- hp adaptivity

Shortfalls for the moment

- efficient solution methods ?
- expensive evaluation of the terms due to Gauss quadrature
- viscous terms
- turbulence model
- shock capturing

DGFEM

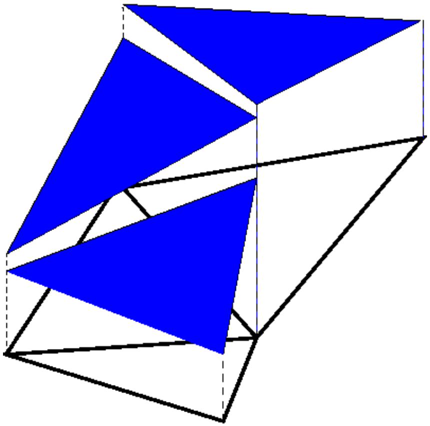
Find the solution u in $\mathcal{D}$ to

$$\mathcal{L}(u) = \nabla \cdot \vec{f}(u) = 0$$

with appropriate boundary conditions.

Approximate solution in broken space $\mathcal{U}$

$$\tilde{u} = \sum_i \mathbf{u}_i \, \phi_i \ , \ \phi_i \in \mathcal{U}$$



Galerkin weighting:

$$\mathbf{r}_j = \int_{\mathcal{D}} \phi_j \frac{\partial \tilde{u}}{\partial t} dV + \int_{\mathcal{D}} \phi_j \nabla \cdot \vec{f}(\tilde{u}) \, dV = 0 \ , \ \forall \phi_j \in \mathcal{U}$$

$$\mathbf{r}_j = \int_T \phi_j \frac{\partial \tilde{u}}{\partial t} dV + \oint_{\partial T} \phi_j \vec{f}(\tilde{u}) \cdot \vec{n} dS - \int_T \nabla \phi_j \cdot \vec{f}(\tilde{u}) \, dV$$

$$= \int_T \phi_j \frac{\partial \tilde{u}}{\partial t} dV + \oint_{\partial T} \phi_j \mathcal{H}\left(\tilde{u}^+, \tilde{u}^-, \vec{n}\right) dS - \int_T \nabla \phi_j \cdot \vec{f}(\tilde{u}) \, dV$$

Solvers

- Runge-Kutta local timestepping

$$\Delta\tau = CFL \cdot \frac{L}{u} \cdot \frac{1}{2p+1}$$

- Newton-Krylov with ILU preconditioner
- Multigrid
  - mesh resolution : h-Multigrid
  - order of solution : p-Multigrid

Newton-Krylov-ILU

- Nonlinear iteration : Newton solution for damped equations

$$\mathbf{r}^*_j = \left(\frac{\tilde{u}^n - \tilde{u}^{n-1}}{\Delta\tau^n} + \nabla \cdot \vec{f}\left(\tilde{u}^n\right), \phi_i \right) = 0 \ , \ \forall \phi_i \in \mathcal{U}$$

resulting in

$$\left(\frac{\mathbf{M}}{\Delta\tau^n} + \frac{\partial \mathbf{r}}{\partial \mathbf{u}} \right) \cdot \Delta\mathbf{u}^n = -\mathbf{r}\left(\mathbf{u}^{n-1}\right)$$

- Linear iterations : GMRES
  - “Matrix-free” implementation

$$\left(\frac{\mathbf{M}}{\Delta\tau^n} + \frac{\partial \mathbf{r}}{\partial \mathbf{u}} \right) \cdot \mathbf{s} \approx \frac{\mathbf{r}^*\left(\mathbf{u}^n + \epsilon \cdot \mathbf{s}\right) - \mathbf{r}^*\left(\mathbf{u}^n\right)}{\epsilon}$$

  - block ILU preconditioner : one block per element, two per face
    - 2D : dofs/element $\sim p^2$ hence elementary matrix $\sim p^4$
    - 3D : dofs/element $\sim p^3$ hence elementary matrix $\sim p^6$

Multigrid

- MG uses successively coarser representations of the solution to iterate on
  - grid size: h-Multigrid (Bastian et al. 2000, Van der Vegt et al. 2002)
  - interpolation order: p-Multigrid (Helenbrook et al. 2003, Fidkowski et al. 2004)
- MG has optimal complexity (TME): ie. work needed to solve the system is only proportional to the number of dofs.
- MG convergence hinges upon
  - fast reduction of high-frequency errors - simple smoothers suffice
  - good quality of transfer operators
- MG is extremely easy to implement in a discontinuous interpolation framework

Multigrid : FAS for Galerkin FEM

"Fine" solution $\tilde{u}^p \in \mathcal{U}^p$

"Coarse" solution $\tilde{u}^q \in \mathcal{U}^q$

- presmoothing on level p, leading to solution $\tilde{u}^{p*}$
- restriction

$$\tilde{u}^{q*} = \mathcal{T}^{qp}\left(\tilde{u}^{p*}\right)$$

- defect correction

$$\left(\mathcal{L}\left(\tilde{u}^q\right) - \mathcal{L}\left(\tilde{u}^{q*}\right) + \mathcal{L}\left(\tilde{u}^{p*}\right), v^q\right) = 0 \ \ \forall v^q \in \mathcal{U}^q$$

- prolongation

$$\tilde{u}^p = \tilde{u}^{p*} + \mathcal{T}^{pq}\left(\tilde{u}^q - \tilde{u}^{q*}\right)$$

- post-smoothing: perform additional iterations on level p to smooth the corrected solution

Multigrid : Solution transfer

$\mathcal{T}^{ba}$ is defined as the L_2 projection from $u^a \in \mathcal{U}^a$ to $u^b \in \mathcal{U}^b$:

$$\sum_i \left(\phi_k^b, \phi_j^b\right) \mathbf{u}_j^b = \sum_j \left(\phi_k^b, \phi_i^a\right) \mathbf{u}_i^a, \ \forall \phi_k^b \in \mathcal{U}^b$$

Equivalent discrete operator

$$\mathbf{u}^b = \mathbf{T}^{ba} \cdot \mathbf{u}^a$$

$$\mathbf{T}^{ba} = \left(\mathbf{M}^{bb}\right)^{-1} \cdot \mathbf{M}^{ba}$$

where

$$\mathbf{M}_{ij}^{bb} = \left(\phi_i^b, \phi_j^b\right)$$

$$\mathbf{M}_{ij}^{ba} = \left(\phi_i^b, \phi_j^a\right)$$

Multigrid : Residual restriction

Avoid modified residual operators associated to mixed weighting

- project residual function on $\mathcal{U}^p$ using L_2 projection:

$$\mathcal{L}\left(\tilde{u}^p\right) \approx \sum_i \mathbf{l}_i^p \phi_i^p$$

$$\sum_i \mathbf{l}_i^p \left(\phi_i^p, \phi_j^p\right) \approx \left(\mathcal{L}\left(\tilde{u}^p\right), \phi_j^p\right) = \mathbf{r}_j^p$$

- then Galerkin weighting becomes trivial:

$$\left(\mathcal{L}\left(\tilde{u}^p\right), \phi_j^q\right) \approx \sum_i \mathbf{l}_i^p \left(\phi_i^p, \phi_j^q\right)$$

Equivalent discrete operator:

$$\mathbf{r}^q = \tilde{\mathbf{T}}^{qp} \cdot \mathbf{r}^p$$

$$\tilde{\mathbf{T}}^{qp} = \mathbf{M}^{qp} \cdot \left(\mathbf{M}^p\right)^{-1} = \left(\mathbf{T}^{pq}\right)^T$$

Multigrid : convergence analysis

Suppose $\mathcal{L}\left(\right)$ is a linear operator, then

$$\left(\mathcal{L}\left(\tilde{u}^p\right), \phi_i^p\right) = \left(f, \phi_i^p\right) \;,\; \forall \phi_i^p \in \mathcal{U}^p$$

is equivalent to

$$\mathbf{L}^p \cdot \mathbf{u}^p = \mathbf{f}^p$$

$$\mathbf{L}_{ij}^p = \left(\phi_i^p, \mathcal{L}\left(\phi_j^p\right)\right)$$

Suppose that interpolation space $\mathcal{U}^q$ is “embedded” in $\mathcal{U}^p$

$$\phi_i^q = \alpha_{ij}^{qp} \cdot \phi_j^p \;,\; \forall \phi_i^q \in \mathcal{U}^q$$

then

$$\alpha^{qp} = \left(\mathbf{M}^{qp} \cdot \left(\mathbf{M}^{pp}\right)^{-1}\right) = \tilde{\mathbf{T}}^{qp} = \left(\mathbf{T}^{pq}\right)^T$$

Multigrid : convergence analysis

We can prove in this case

$$\begin{aligned}\mathbf{L}^q_{ij} &= \left(\phi^q_i, \mathcal{L}\left(\phi^q_j\right)\right) \\ &= \alpha^{qp}_{ik} \cdot \left(\phi^p_k, \mathcal{L}\left(\phi^p_l\right)\right) \cdot \alpha^{qp}_{jl}\end{aligned}$$

Hence for the chosen transfer operators the discrete coarse grid approximation (**DCGA**) coincides with the Galerkin coarse grid approximation (**GCGA**) :

$$\mathbf{L}^q = \tilde{\mathbf{T}}^{qp} \cdot \mathbf{L}^p \cdot \mathbf{T}^{pq}$$

Hence the discretised defect correction equation

$$\mathbf{L}^q \cdot \left(\mathbf{u}^q - \mathbf{u}^{q*}\right) + \tilde{\mathbf{T}}^{qp} \cdot \left(\mathbf{L}^p \cdot \mathbf{u}^{p*} - \mathbf{f}^p\right) = 0$$

translates to:

$$\tilde{\mathbf{T}}^{qp}\left(\mathbf{L}^p \cdot \left(\mathbf{u}^{p^*} + \mathbf{T}^{pq} \cdot \left(\mathbf{u}^q - \mathbf{u}^{q*}\right)\right) - \mathbf{f}^p\right) = 0$$

Multigrid : convergence analysis

We define the error $\mathbf{e}^p$ as:

$$\mathbf{L}^p \mathbf{e}^p = \mathbf{L}^p \mathbf{u}^p - \mathbf{f}^p = \mathbf{r}^p$$

The error $\mathbf{e}^p$ after one two-level iteration now becomes

$$\mathbf{e}^p = \left(\mathbf{I}p - \mathbf{T}^{pq} \cdot \left(\mathbf{L}^q \right)^{-1} \cdot \tilde{\mathbf{T}}^{qp} \cdot \mathbf{L}^p \right) \cdot \mathbf{e}^{p*}$$

We may now decompose the error $\mathbf{e}^{p*}$

$$\mathbf{e}^{p*} = \mathbf{e}_s^{p*} + \mathbf{e}_r^{p*}$$

$$\mathbf{e}_s^{p*} = \left(\mathbf{T}^{pq} \cdot \left(\tilde{\mathbf{T}}^{qp} \mathbf{T}^{pq} \right)^{-1} \cdot \tilde{\mathbf{T}}^{qp} \right) \cdot \mathbf{e}^{p*}$$

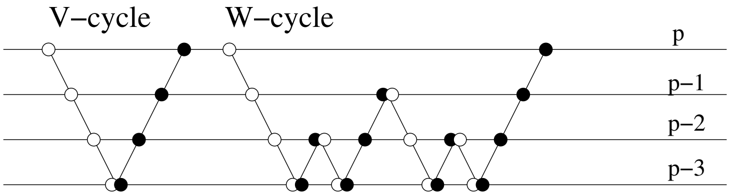
$$\mathbf{e}_r^{p*} = \left(\mathbf{I}p - \mathbf{T}^{pq} \cdot \left(\tilde{\mathbf{T}}^{qp} \mathbf{T}^{pq} \right)^{-1} \cdot \tilde{\mathbf{T}}^{qp} \right) \cdot \mathbf{e}^{p*}$$

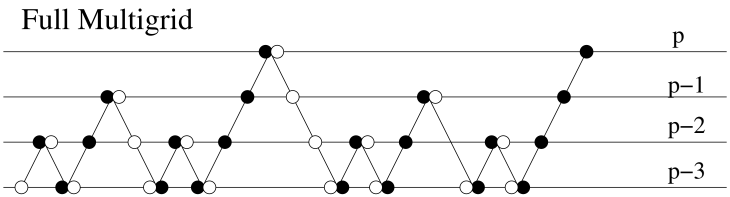
after the coarse grid correction we find

$$\mathbf{e}^p = \left(\mathbf{I}p - \mathbf{T}^{pq} \cdot \left(\mathbf{L}^q \right)^{-1} \cdot \tilde{\mathbf{T}}^{qp} \cdot \mathbf{L}^p \right) \cdot \mathbf{e}_r^{p*}$$

Multigrid strategy

- cycling

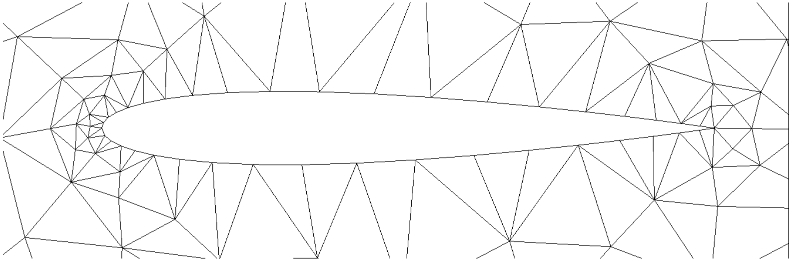


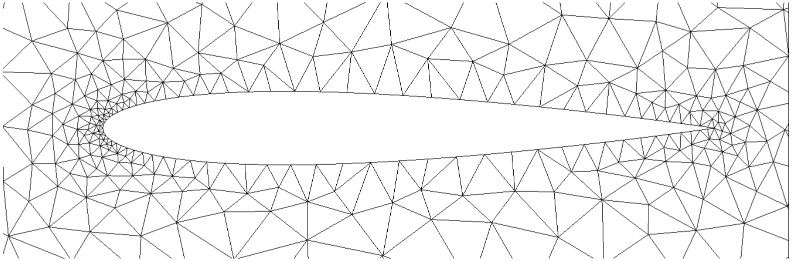


- solvers and solver parameters for each level
  - explicit methods on fine levels
  - implicit methods on coarse levels

NACA0012 airfoil

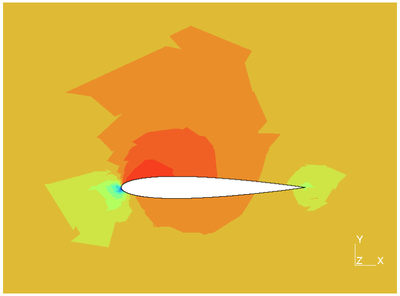
$M = 0.3,\, \alpha = 2^o$



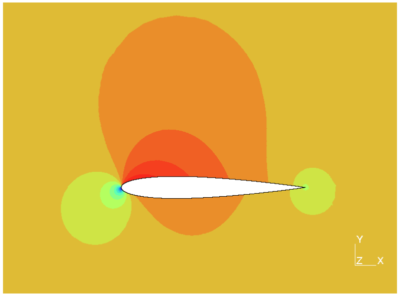


NACA0012: Mach isolines

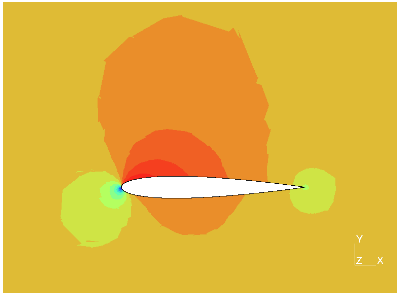
p=1, coarse



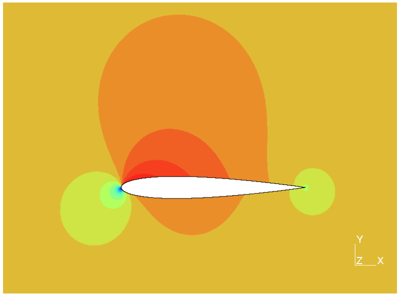
p=4, coarse



p=1, fine

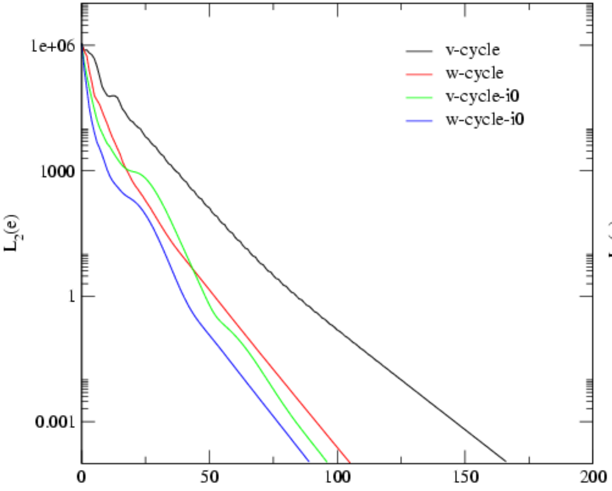


p=4, fine

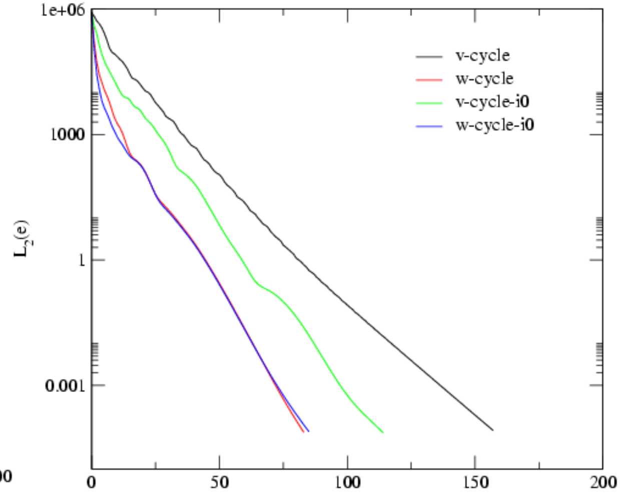


NACA0012 : strategies compared (cycles,coarse)

order 2

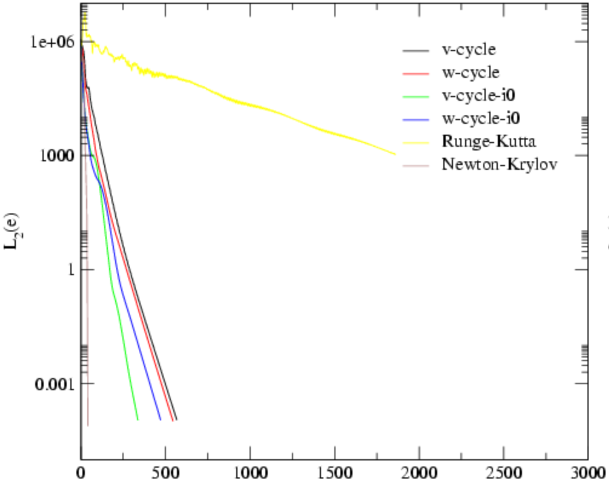


order 4

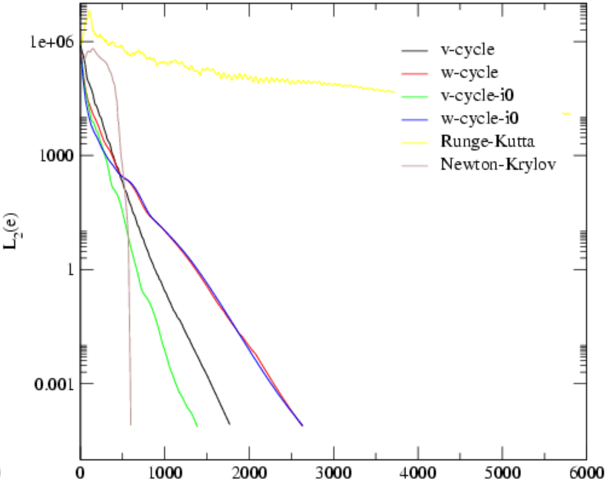


NACA0012 : strategies compared (CPU,coarse)

order 2

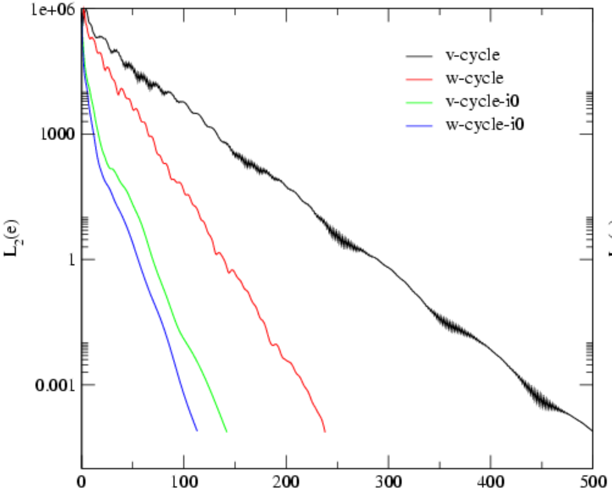


order 4

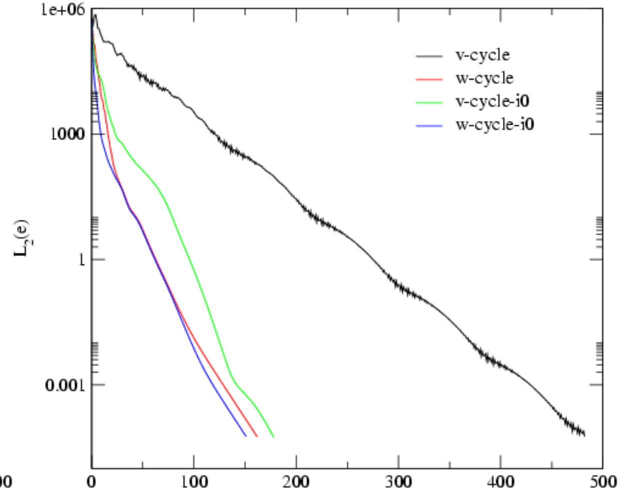


NACA0012 : strategies compared (cycles,fine)

order 2

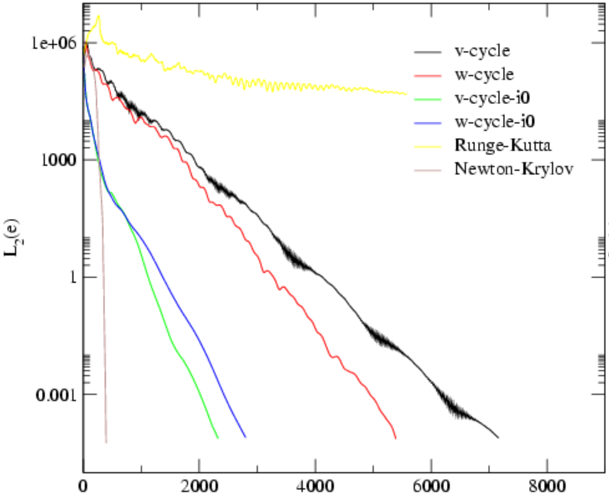


order 4

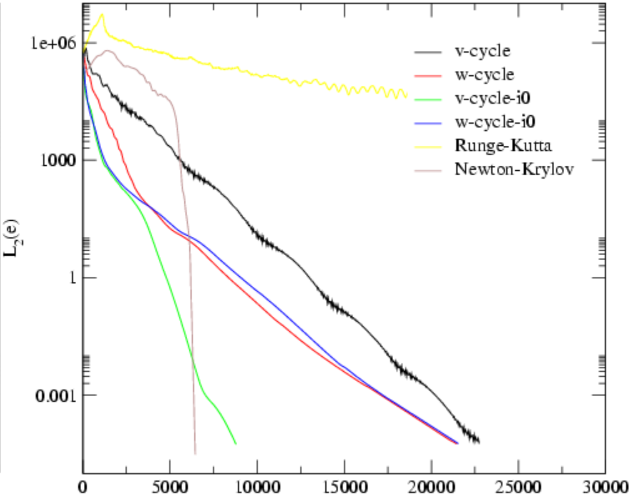


NACA0012 : strategies compared (CPU,fine)

order 2

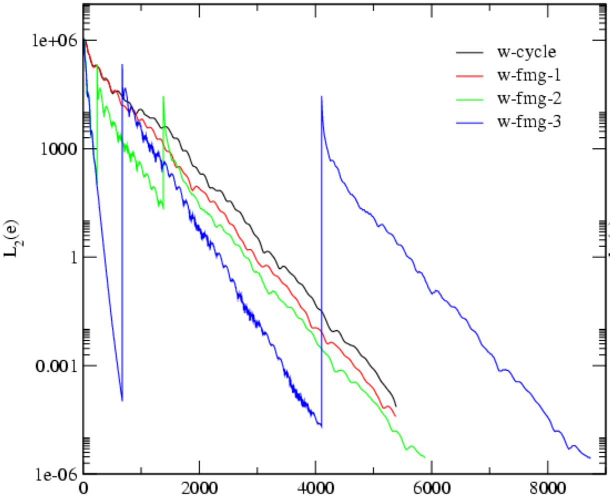


order 4

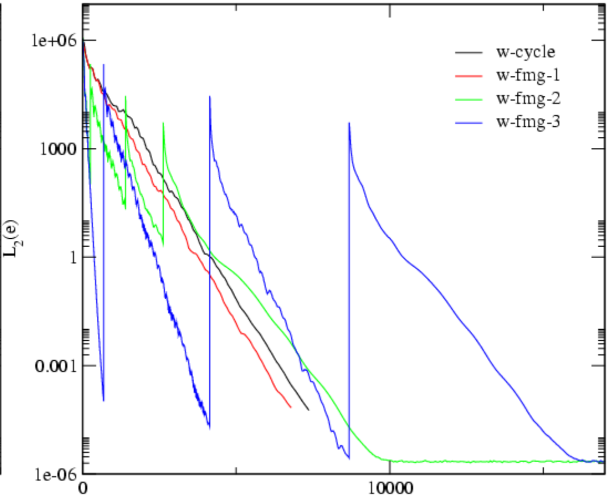


NACA0012 : strategies compared (FMG)

order 2, fine mesh



order 3, fine mesh



Conclusions and future work

Conclusions:

- generic framework for multigrid in DGFEM (equally applicable to h-multigrid on conforming or non-conforming meshes and non-hierarchic p-multigrid);
- p-MG convergence independent of order (TME), but dependent on grid size;
- using pure explicit smoothers, w-cycle is most performant;
- using implicit coarsest level v-cycle provides significant speed-ups, providing viable alternative to Newton-Krylov-ILU(0);
- FMG does not improve convergence rates.

Future work:

- look for a more performant smoother for higher order levels;
- combine with h-multigrid to obtain h- and p-independent convergence rates;
- use MG as preconditioner for GMRES;
- directional p-coarsening in boundary layers.

TOC