COMPUTATION OF KRONECKER-LIKE FORMS OF A SYSTEM PENCIL: APPLICATIONS, ALGORITHMS AND SOFTWARE

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Abstract

Kronecker-like forms of a system pencil are useful in solving many computational problems encountered in the analysis and synthesis of linear systems. The reduction of system pencils to various Kronecker-like forms can be performed by structure preserving $0(n^3)$ complexity numerically stable algorithms. The presented algorithms form the basis of a modular collection of LAPACK compatible FORTRAN 77 subroutines to perform the reduction of a system pencil to several Kronecker-like forms.

1. Introduction

The most general representation of a linear time-invariant system is the *generalized state space* or *descriptor* model

$$\begin{array}{rcl} \lambda \boldsymbol{E}\boldsymbol{x}(t) & = & \boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) & = & \boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}\boldsymbol{u}(t) \end{array} \qquad (1)$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the descriptor state vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the input vector and $\boldsymbol{y}(t) \in \mathbb{R}^p$ is the output vector, and where $\boldsymbol{A} \in \mathbb{R}^{\ell\times n}$, $\boldsymbol{E} \in \mathbb{R}^{\ell\times n}$, $\boldsymbol{B} \in \mathbb{R}^{\ell\times m}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $\boldsymbol{D} \in \mathbb{R}^{p\times m}$. Notice that generally $\boldsymbol{A}$ and $\boldsymbol{E}$ are non-square matrices and even if these matrices are square, $\boldsymbol{E}$ may be singular. In the case of *standard systems* $\boldsymbol{E}$ is an invertible matrix and in most of cases $\boldsymbol{E} = \boldsymbol{I}_n$, the n-th order identity matrix. The operator λ is either the differential operator $\lambda\boldsymbol{x}(t) = d\boldsymbol{x}(t)/dt$ or the advance operator $\lambda \boldsymbol{x}(t) = \boldsymbol{x}(t+1)$. We denote alternatively the system (1) by the quadruple $(\boldsymbol{A}-\lambda\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$.

For this representation one can define the associated *system matrix* of (1) as the linear pencil

$$\mathcal{S}(\lambda) = \left[\begin{array}{c|c} \boldsymbol{B} & \boldsymbol{A}-\lambda\boldsymbol{E} \\ \hline \boldsymbol{D} & \boldsymbol{C} \end{array}\right]. \qquad (2)$$

The *Kronecker's canonical form* (KCF) [1] of this pencil can be obtained by applying to $\mathcal{S}(\lambda)$ suitable invertible left and right transformations $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively, to yield a block diagonal decomposition of the form

$$\boldsymbol{U}\mathcal{S}(\lambda)\boldsymbol{V} = \text{diag}\,\{\boldsymbol{L}_\epsilon, \boldsymbol{I}-\lambda\boldsymbol{J}_\infty, \boldsymbol{J}_f-\lambda I, \boldsymbol{L}_\eta^T\} \qquad (3)$$

where: (a) $\boldsymbol{L}_\epsilon = \text{diag}\,\{\boldsymbol{L}_{\epsilon_1}, \ldots, \boldsymbol{L}_{\epsilon_q}\}$, $\boldsymbol{L}_\eta^T = \text{diag}\,\{\boldsymbol{L}_{\eta_1}^T, \ldots, \boldsymbol{L}_{\eta_r}^T\}$, and $\boldsymbol{L}_i$ is the $i\times(i+1)$ bidiagonal pencil

$$\boldsymbol{L}_i = \left[\begin{array}{cccc} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{array}\right] \qquad (4)$$

(b) $\boldsymbol{J}_\infty = \text{diag}\,\{\boldsymbol{J}_{\nu_1}(0), \ldots, \boldsymbol{J}_{\nu_s}(0)\}$ and $\boldsymbol{J}_i(0)$ is the Jordan block of order i corresponding to the null eigenvalue. Notice that the matrix $\boldsymbol{J}_\infty$ is nilpotent.

(c) $\boldsymbol{J}_f$ is a matrix in Jordan canonical form.

The pencils $\boldsymbol{J}_f - \lambda I$ and $I-\lambda\boldsymbol{J}_\infty$ contain the finite and infinite eigenvalues, respectively, and represent together the *regular* part of $\mathcal{S}(\lambda)$. The finite eigenvalues are also called the *finite zeros* of $\mathcal{S}(\lambda)$. To each Jordan block $\boldsymbol{J}_{\nu_i}(0)$ corresponds an infinite elementary divisor of order $\nu_i - 1$ in the Schmidt form of the polynomial matrix $\mathcal{S}(\lambda)$, and thus the union of the sets of $\nu_i - 1$ infinite eigenvalues is also called the *infinite zeros* of $\mathcal{S}(\lambda)$. The blocks $\boldsymbol{L}_\epsilon$ and $\boldsymbol{L}_\eta^T$ contain the *singularity* of $\mathcal{S}(\lambda)$ and the index sets $\{\epsilon_i\}$ and $\{\eta_i\}$ are the *left* (or *column*) and *right* (or *row*) *minimal Kronecker indices* of $\mathcal{S}(\lambda)$, respectively. Notice that zero row or column indices correspond to null rows or null columns in the KCF of the system pencil, respectively.

The KCF is very useful in the structural analysis of descriptor systems. Particular system matrices can be used to study the pole-zeros structure or the controllability-observability properties of a system by computing various type of zeros (poles, input or output decoupling zeros) [2]. Controllability and observability indices can be easily deduced from the row and column Kronecker indices of the particular system pencils

$[\ \boldsymbol{B} \mid \boldsymbol{A}-\lambda\boldsymbol{E}\ ]$ and $\left[\begin{array}{c}\boldsymbol{A}-\lambda\boldsymbol{E} \\ \hline \boldsymbol{C}\end{array}\right]$. The KCF also provides information on the left and right null-space structure of $\mathcal{S}(\lambda)$ and generalized inverses of the system pencil can be computed using this information, having as main application the inversion of rational matrices [3]. The need to determine generalized inverses of rational matrices arises in some of recently developed algorithms to compute minimum-phase rational coprime factorizations, as for example the *inner-outer* factorization [4] or the *J-inner-outer* factorization [5]. The reduction of special system pencils to KCF can also be used to solve special classes of constrained Riccati equations [6, 7].

In all above applications the computation of the KCF is in fact not necessary and certainly not recommendable from numerical point of view. Instead, with the help of orthogonal left and right transformations, several condensed Kronecker-like forms can be computed which exhibit either the complete Kronecker structure or only a part of the Kronecker structure of the system pencil. In the next sections we introduce several Kronecker-like forms which can be computed with the help of a collection of recently implemented FORTRAN 77 subroutines. For an easy reference, we shall associate each form with the name of the corresponding subroutine implemented to compute it. We also indicate the main applicability of each form in solving some of the above mentioned problems.

The algorithms to compute various Kronecker-like forms are combinations of several recently developed numerically stable procedures [8, 9, 10, 2]. All implemented algorithms have $0(n^3)$ computational complexity and compare favorably in many aspects with existing methods [11, 8, 12]. Because of space restrictions we will discuss only three Kronecker-like forms. Other condensed forms and the accompanying software are discussed in an extended version of this paper [13].

2. The SPRED Form

The SPRED subroutine determines, by applying left and right orthogonal transformations, the following Kronecker-like form of the system pencil

$$\widehat{\mathcal{S}}(\lambda) = \boldsymbol{Q}^T\mathcal{S}(\lambda)\boldsymbol{Z} =$$

$$\left[\begin{array}{c|ccccc} \boldsymbol{B}_r & \boldsymbol{A}_r-\lambda\boldsymbol{E}_r & * & * & * & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_\infty-\lambda\boldsymbol{E}_\infty & * & * & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{D}_i & * & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_f-\lambda\boldsymbol{E}_f & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_l-\lambda\boldsymbol{E}_l \\ \hline \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C}_l \end{array}\right] \tag{5}$$

where: (a) the pencil $[\ \boldsymbol{B}_r\ \boldsymbol{A}_r-\lambda\boldsymbol{E}_r\ ]$ contains the right Kronecker structure of $\mathcal{S}(\lambda)$ and $\boldsymbol{E}_r$ is invertible and upper-triangular; the pair $(\boldsymbol{B}_r, \boldsymbol{A}_r-\lambda\boldsymbol{E}_r)$ is controllable and the pencil $[\ \boldsymbol{B}_r\ \boldsymbol{A}_r-\lambda\boldsymbol{E}_r\ ]$ is in the controllability staircase form.

(b) the *regular* pencil $\boldsymbol{A}_\infty-\lambda\boldsymbol{E}_\infty$ together with $\boldsymbol{D}_i$ contain the infinity Kronecker structure of $\mathcal{S}(\lambda)$; $\boldsymbol{A}_\infty$ and $\boldsymbol{D}_i$ are invertible and upper-triangular, and $\boldsymbol{E}_\infty$ is nilpotent and upper-triangular.

(c) the *regular* pencil $\boldsymbol{A}_f-\lambda\boldsymbol{E}_f$ contains the finite Kronecker structure of $\mathcal{S}(\lambda)$ and $\boldsymbol{E}_f$ is invertible and upper-triangular.

(d) the pencil $\left[\begin{array}{c}\boldsymbol{A}_l-\lambda\boldsymbol{E}_l \\ \boldsymbol{C}_l\end{array}\right]$ contains the left Kronecker structure of $\mathcal{S}(\lambda)$ and $\boldsymbol{E}_l$ is invertible and upper-triangular; the pair $(\boldsymbol{C}_l, \boldsymbol{A}_l-\lambda\boldsymbol{E}_l)$ is observable and the pencil $\left[\begin{array}{c}\boldsymbol{A}_l-\lambda\boldsymbol{E}_l \\ \boldsymbol{C}_l\end{array}\right]$ is in the observability staircase form.

Excepting the finite eigenvalues structure, the SPRED form contains identical structural information as the KCF. The detailed structure of the subpencils of the SPRED form is given in [13]. The associated dimensional index sets determine the minimal indices and the infinite structure of the system pencil $\mathcal{S}(\lambda)$.

Application. Let $\boldsymbol{G}(\lambda)$ be a $p\times m$ rational matrix for which we want to compute a (1,2)-*generalized inverse* $\boldsymbol{G}(\lambda)^+$ satisfying the conditions $\boldsymbol{G}\boldsymbol{G}^+\boldsymbol{G}=\boldsymbol{G}$ and $\boldsymbol{G}^+\boldsymbol{G}\boldsymbol{G}^+=\boldsymbol{G}^+$ [14]. Each $\boldsymbol{G}(\lambda)$ can be assimilated with the *transfer-function matrix* (TFM) of a *regular* descriptor system $(\boldsymbol{A}-\lambda\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$, satisfying

$$\boldsymbol{G}(\lambda) = \boldsymbol{C}(\lambda\boldsymbol{E}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}. \tag{6}$$

The generalized inverse $\boldsymbol{G}(\lambda)^+$ of $\boldsymbol{G}(\lambda)$ can be computed by using the formula [3]

$$\boldsymbol{G}(\lambda)^+ = [\ \boldsymbol{O}\ \boldsymbol{I}_m\ ]\ \mathcal{S}(\lambda)^+ \left[\begin{array}{c}\boldsymbol{I}_p \\ \boldsymbol{O}\end{array}\right]. \tag{7}$$

With the partitioning of $\widehat{\mathcal{S}}(\lambda)$ in (5) as

$$\widehat{\mathcal{S}}(\lambda) = \left[\begin{array}{c|c}\widehat{\mathcal{S}}_{11}(\lambda) & \widehat{\mathcal{S}}_{12}(\lambda) \\ \hline \boldsymbol{O} & \widehat{\mathcal{S}}_{22}(\lambda)\end{array}\right] \tag{8}$$

it follows that for almost all λ, $\operatorname{rank}\mathcal{S}(\lambda) = \operatorname{rank}\widehat{\mathcal{S}}_{12}(\lambda)$, and thus a generalized (1,2)-inverse of $\mathcal{S}(\lambda)$ can be computed as [14]

$$\mathcal{S}(\lambda)^+ = \boldsymbol{Z}\left[\begin{array}{cc}\boldsymbol{O} & \boldsymbol{O} \\ \widehat{\mathcal{S}}_{12}(\lambda)^{-1} & \boldsymbol{O}\end{array}\right]\boldsymbol{Q}^T. \tag{9}$$

It is easy to verify that $G(\lambda)^+$ in (7) is indeed an (1,2)-generalized inverse of $\boldsymbol{G}(\lambda)$. Notice that to compute a descriptor representation of the generalized inverse $\boldsymbol{G}(\lambda)^+$, it is not necessary to explicitly evaluate $\widehat{\mathcal{S}}_{12}(\lambda)^{-1}$ [3].

The finite pole structure of the generalized inverse $\boldsymbol{G}^{+}$ results from the SPRED form (5) of the system pencil $\mathcal{S}(\lambda)$ used to compute it. Thus, the finite poles of $\boldsymbol{G}^{+}$ are the union of generalized eigenvalues of the pair $(\boldsymbol{A}_f, \boldsymbol{E}_f)$ called also the *zeros* of $\boldsymbol{G}$ and of the generalized eigenvalues of the pairs $(\boldsymbol{A}_r, \boldsymbol{E}_r)$ and $(\boldsymbol{A}_l, \boldsymbol{E}_l)$. Notice that the zeros of $\boldsymbol{G}$ are always present among the poles of any of its generalized inverses. Even if the zeros are stable, that is the descriptor system is *minimum-phase*, it is still possible that the generalized inverse has unstable poles because of possible unstable eigenvalues appearing in the pairs $(\boldsymbol{A}_r, \boldsymbol{E}_r)$ and $(\boldsymbol{A}_l, \boldsymbol{E}_l)$. However the spectrums of these pairs can be arbitrarily modified by applying suitable left and right non-orthogonal transformations to the SPRED form (5). Thus stable inverses can be computed provided the given system is minimum-phase. The computation of inverses is further discussed in [3].

3. The SLRRED Form

The SLRRED subroutine determines, by applying left and right orthogonal transformations, the following Kronecker-like form of the system pencil

$$\widehat{\mathcal{S}}(\lambda) = \boldsymbol{Q}^T \mathcal{S}(\lambda) \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{A}_r - \lambda \boldsymbol{E}_r & * & * & * \\ \boldsymbol{O} & \boldsymbol{D}_i & * & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_f - \lambda \boldsymbol{E}_f & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_l - \lambda \boldsymbol{E}_l \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C}_l \end{bmatrix},$$

where: (a) $\boldsymbol{A}_r - \lambda \boldsymbol{E}_r$ has full row rank and contains the right and infinite Kronecker structure of $\mathcal{S}(\lambda)$; the pencil $\boldsymbol{A}_r - \lambda \boldsymbol{E}_r$ is in a staircase form.

(b) The rest of subpencils are as in the SPRED form.

Excepting the finite eigenvalues structure, the SLRRED form contains identical structural information as the KCF. The detailed structure of the subpencil $\boldsymbol{A}_r - \lambda \boldsymbol{E}_r$ is given in [13]. The associated dimensional index sets determine the right minimal indices and the infinite structure of the system pencil $\mathcal{S}(\lambda)$.

Application. It is well known that each stable rational matrix $\boldsymbol{G}(\lambda)$ has an *inner-outer factorization* $\boldsymbol{G} = \boldsymbol{G}_i \boldsymbol{G}_o$, where $\boldsymbol{G}_i$ is a *square* inner factor and $\boldsymbol{G}_o$ is a stable and minimum-phase TFM. The main computational problem in the procedure proposed in [4] is the computation of $\boldsymbol{G}_i$ from the right-coprime factorization with inner denominator of a particular (1,2)-generalized inverse $\boldsymbol{G}^{+}$ of $\boldsymbol{G}$ as $\boldsymbol{G}^{+} = \boldsymbol{N}\boldsymbol{G}_i^{-1}$. The outer factor results simply as $\boldsymbol{G}_o = \boldsymbol{G}_i^{-1}\boldsymbol{G}$. For the computation of the inner denominator a recursive state-space algorithm described also in [4] is best suited. By using this algorithm to compute $\boldsymbol{G}_i$ the output matrices of the descriptor realization of $\boldsymbol{G}^{+}$ play no role. Thus instead of the SPRED form to compute an appropriate $\boldsymbol{G}(\lambda)^{+}$, we can use the simpler SLRRED form, which provides all necessary information to compute $\boldsymbol{G}_i$. A detailed description of the resulting algorithm is presented in [4]. A similar technique can be used to compute J-inner-outer factorizations [5].

4. The SRLRED Form

The SRLRED subroutine determines, by applying left and right orthogonal transformations, the following Kronecker-like form of the system pencil

$$\widehat{\mathcal{S}}(\lambda) = \boldsymbol{Q}^T \mathcal{S}(\lambda) \boldsymbol{Z} =$$

$$\begin{bmatrix} \boldsymbol{B}_r & \boldsymbol{A}_r - \lambda \boldsymbol{E}_r & * & * & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_f - \lambda \boldsymbol{E}_f & * & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{D}_i & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_l - \lambda \boldsymbol{E}_l \end{bmatrix}, \qquad (10)$$

where: (a) $\boldsymbol{A}_l - \lambda \boldsymbol{E}_l$ has full column rank and contains the left and infinite Kronecker structure of $\mathcal{S}(\lambda)$; the pencil $\boldsymbol{A}_l - \lambda \boldsymbol{E}_l$ is in the staircase form

(b) The rest of subpencils are as in the SPRED form.

Excepting the finite eigenvalues structure, the SRLRED form contains identical structural information as the KCF. The detailed structure of the subpencil $\boldsymbol{A}_l - \lambda \boldsymbol{E}_l$ is given in [13]. The associated dimensional index sets determine the left minimal indices and the infinite structure of the system pencil $\mathcal{S}(\lambda)$.

Application. The computation of *maximal proper stable deflating subspaces* of Hamiltonian and simplectic pencils has important applications in solving various nonstandard Riccati equations [6, 7]. In what follows, we discuss how to compute such subspaces for an arbitrary pencil by using the SRLRED form. Let $\boldsymbol{M} - \lambda \boldsymbol{N}$ be an arbitrary pencil with $\boldsymbol{M}, \boldsymbol{N} \in \mathbb{R}^{r \times q}$, let $\mathbb{C}^-$ be the stability region of $\mathbb{C}$ and let $\mathbb{C}^+$ be the complement of $\mathbb{C}^-$ in $\mathbb{C}$. The following particular reducing subspace introduced in [6] has important applications in solving various nonstandard Riccati equations.

Definition 1 *A subspace $\mathcal{V} \subset \mathbb{R}^q$ of dimension ρ is called a proper stable deflating subspace of $\boldsymbol{M} - \lambda \boldsymbol{N}$ to the right if $\boldsymbol{N}\boldsymbol{V} = \boldsymbol{M}\boldsymbol{V}\boldsymbol{S}$ and $\boldsymbol{M}\boldsymbol{V}$ is monic, where $\boldsymbol{V} \in \mathbb{R}^{q \times \rho}$ is any basis matrix for $\mathcal{V}$ and $\boldsymbol{S} \in \mathbb{R}^{\rho \times \rho}$ is an adequate matrix having all its eigenvalues in $\mathbb{C}^-$.*

Let n_f^- be the number of stable generalized eigenvalues of the pair $(\boldsymbol{A}_f, \boldsymbol{E}_f)$ in the SRLRED form of the pencil $\boldsymbol{M} - \lambda \boldsymbol{N}$ (viewed as a particular system pencil) and let n_r the dimension of the $\boldsymbol{A}_r - \lambda \boldsymbol{E}_r$ block. The following result [6] characterizes the existence of a stable proper deflating subspace of maximal dimension.

Theorem 1 *The pencil $\boldsymbol{M} - \lambda \boldsymbol{N}$ has a stable proper deflating subspace to the right if and only if $n_r + n_f^- > 0$. Moreover, the maximal dimension of a stable proper deflating subspace to the right is $n_r + n_f^-$.*

A procedure (similar to that proposed in [15]) to compute the basis matrix $\boldsymbol{V}$ for a proper deflating subspace of maximal dimension of $\boldsymbol{M} - \lambda\boldsymbol{N}$ has the following main steps:

1. Compute the orthogonal matrices $\boldsymbol{Q}$ and $\boldsymbol{Z}$ to reduce the pencil $\boldsymbol{M} - \lambda\boldsymbol{N}$ to the `SRLRED` form (10).
2. Apply the pole assignment algorithm of [16] to determine the orthogonal matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Z}_1$ and the feedback matrix $\boldsymbol{F}$ such that $\Lambda(\boldsymbol{Q}_1^T(\boldsymbol{A}_r + \boldsymbol{B}_r\boldsymbol{F})\boldsymbol{Z}_1, \boldsymbol{Q}_1^T\boldsymbol{E}_r\boldsymbol{Z}_1) \subset \mathbb{C}^-$ and the pair $(\boldsymbol{Q}_1^T(\boldsymbol{A}_r + \boldsymbol{B}_r\boldsymbol{F})\boldsymbol{Z}_1, \boldsymbol{Q}_1^T\boldsymbol{E}_r\boldsymbol{Z}_1)$ is in a *generalized real Schur form* (GRSF).
3. Compute the orthogonal matrices $\boldsymbol{Q}_2$ and $\boldsymbol{Z}_2$ to reduce the pair $(\boldsymbol{A}_f, \boldsymbol{E}_f)$ to the ordered GRSF
$$\boldsymbol{Q}_2^T\boldsymbol{A}_f\boldsymbol{Z}_2 = \begin{bmatrix} \boldsymbol{A}_{11}^f & \boldsymbol{A}_{12}^f \\ 0 & \boldsymbol{A}_{22}^f \end{bmatrix}, \ \boldsymbol{Q}_2^T\boldsymbol{E}_f\boldsymbol{Z}_2 = \begin{bmatrix} \boldsymbol{E}_{11}^f & \boldsymbol{E}_{12}^f \\ 0 & \boldsymbol{E}_{22}^f \end{bmatrix},$$
where $\Lambda(\boldsymbol{A}_{11}^f, \boldsymbol{E}_{11}^f) \subset \mathbb{C}^-$ and $\Lambda(\boldsymbol{A}_{22}^f, \boldsymbol{E}_{22}^f) \subset \mathbb{C}^+$.
4. Compute $\boldsymbol{V}$ as
$$\boldsymbol{V} = \boldsymbol{Z}\begin{bmatrix} \boldsymbol{F} & \boldsymbol{O} \\ \boldsymbol{Z}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{Z}_2 \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}$$

Notice that in the above procedure, the left transformations should be not accumulated. It is easy to verify that $\boldsymbol{NV} = \boldsymbol{MVS}$ holds with $\boldsymbol{MV}$ monic,
$$S := \begin{bmatrix} \boldsymbol{Z}_1^T\boldsymbol{E}_r^{-1}(\boldsymbol{A}_r + \boldsymbol{B}_r\boldsymbol{F})\boldsymbol{Z}_1 & * \\ \boldsymbol{O} & (\boldsymbol{E}_{11}^f)^{-1}\boldsymbol{A}_{11}^f \end{bmatrix},$$
and $\Lambda(\boldsymbol{S}) \subset \mathbb{C}^-$.

The reductions to the ordered GRSF at steps 2 and 3 can be performed by using the well-known QZ algorithm of [17] followed by the recently developed numerically stable algorithms to reorder the GRSF [18].

5. Algorithms

In this section we discuss the computational approaches implemented in the subroutines to compute the Kronecker-like forms introduced in previous sections. A common characteristics of all these procedures is that they consist of combinations of several highly specialized structure revealing subprocedures, as those to separate the left and infinity structures, the right and infinity structures, the controllable and uncontrollable or the observable and unobservable parts of particular system pencils. As an example, we present the main computational steps of the most complex procedure to compute the `SPRED` form. We also discuss the computational ingredients which ensure the $0(n^3)$ computational complexity of this procedure. The procedures to compute other Kronecker-like forms are either parts of the `SPRED` Procedure or rely on similar dual algorithms applied to implicitly pertransposed pencils (transposed with respect to the main antidiagonal).

`SPRED` Procedure.

1. Determine orthogonal $\boldsymbol{U}_1$ and $\boldsymbol{V}_1$ to compute a complete orthogonal decomposition of $\boldsymbol{E}$ in the form $\boldsymbol{E} = \boldsymbol{U}_1\begin{bmatrix} \boldsymbol{O} & \widehat{\boldsymbol{E}} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}\boldsymbol{V}_1^T$, where $\widehat{\boldsymbol{E}}$ is upper-triangular and non-singular. Set $\boldsymbol{Q} = \operatorname{diag}\{\boldsymbol{U}_1, \boldsymbol{I}_p\}$, $\boldsymbol{Z} = \operatorname{diag}\{\boldsymbol{I}_m, \boldsymbol{V}_1\}$ and compute the `SRSET` form of $\mathcal{S}(\lambda)$ as
$$\mathcal{S}_1(\lambda) = \boldsymbol{Q}^T\mathcal{S}(\lambda)\boldsymbol{Z} := \left[\begin{array}{c|c} \widehat{\boldsymbol{B}} & \widehat{\boldsymbol{A}} - \lambda\widehat{\boldsymbol{E}} \\ \hline \widehat{\boldsymbol{D}} & \widehat{\boldsymbol{C}} \end{array}\right].$$
2. By using the dual S-REDUCE algorithm of [2], determine orthogonal $\boldsymbol{U}_2$ and $\boldsymbol{V}_2$ to reduce the system pencil $\mathcal{S}_1(\lambda)$ to the `SLISEP` form
$$\mathcal{S}_2(\lambda) = \boldsymbol{U}_2^T\mathcal{S}_1(\lambda)\boldsymbol{V}_2 = \left[\begin{array}{c|cc} \boldsymbol{A}_1 - \lambda\boldsymbol{E}_1 & \multicolumn{2}{c}{*} \\ \hline \boldsymbol{O} & \boldsymbol{B}_c & \boldsymbol{A}_c - \lambda\boldsymbol{E}_c \\ & \boldsymbol{D}_c & \boldsymbol{C}_c \end{array}\right]$$
where $\boldsymbol{D}_c$ is upper-trapezoidal and has full column rank, $\boldsymbol{E}_c$ is upper-triangular and non-singular, and $\boldsymbol{A}_1 - \lambda\boldsymbol{E}_1$ has full row rank. Compute $\boldsymbol{Q} \leftarrow \boldsymbol{QU}_2$, $\boldsymbol{Z} \leftarrow \boldsymbol{ZV}_2$.
3. By using the reduction technique of [8, pages 33-34], determine orthogonal $\boldsymbol{U}_3$ to compress the rows of the matrix $\begin{bmatrix} \boldsymbol{B}_c \\ \boldsymbol{D}_c \end{bmatrix}$ such that
$$\mathcal{S}_3(\lambda) = \operatorname{diag}\{\boldsymbol{I}, \boldsymbol{U}_3^T\}\mathcal{S}_2(\lambda) = \begin{bmatrix} \boldsymbol{A}_1 - \lambda\boldsymbol{E}_1 & * & * \\ \boldsymbol{O} & \boldsymbol{D}_i & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_2 - \lambda\boldsymbol{E}_2 \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{C}_2 \end{bmatrix}$$
where $\boldsymbol{D}_i$ is upper-triangular and non-singular, $\boldsymbol{C}_2$ is the part of $\boldsymbol{C}_c$ corresponding to the linearly dependent rows of $\boldsymbol{D}_c$, and $\boldsymbol{E}_2$ is upper-triangular and non-singular. Compute $\boldsymbol{Q} \leftarrow \boldsymbol{Q}\operatorname{diag}\{\boldsymbol{I}, \boldsymbol{U}_3\}$.
4. By using the dual of the controllability staircase algorithm of [9], determine orthogonal $\boldsymbol{U}_4$ and $\boldsymbol{V}_4$ to reduce the sub-pencil $\begin{bmatrix} \boldsymbol{A}_2 - \lambda\boldsymbol{E}_2 \\ \boldsymbol{C}_2 \end{bmatrix}$ to the observability staircase form
$$\boldsymbol{U}_4^T\begin{bmatrix} \boldsymbol{A}_2 - \lambda\boldsymbol{E}_2 \\ \boldsymbol{C}_2 \end{bmatrix}\boldsymbol{V}_4 = \begin{bmatrix} \boldsymbol{A}_f - \lambda\boldsymbol{E}_f & * \\ \boldsymbol{O} & \boldsymbol{A}_l - \lambda\boldsymbol{E}_l \\ \boldsymbol{O} & \boldsymbol{C}_l \end{bmatrix},$$
where the pair $(\boldsymbol{C}_l, \boldsymbol{A}_l - \lambda\boldsymbol{E}_l)$ is observable, and both $\boldsymbol{E}_f$ and $\boldsymbol{E}_l$ are upper-triangular and non-singular matrices. Compute the `SLRRED` form
$$\mathcal{S}_4(\lambda) = \operatorname{diag}\{\boldsymbol{I}, \boldsymbol{U}_4^T\}\mathcal{S}_3(\lambda)\operatorname{diag}\{\boldsymbol{I}, \boldsymbol{V}_4\}$$
and $\boldsymbol{Q} \leftarrow \boldsymbol{Q}\operatorname{diag}\{\boldsymbol{I}, \boldsymbol{U}_4\}$, $\boldsymbol{Z} \leftarrow \boldsymbol{Z}\operatorname{diag}\{\boldsymbol{I}, \boldsymbol{V}_4\}$.

5. By using Algorithms 3.3.1 and 3.3.3 in [8], determine orthogonal $\boldsymbol{U}_5$ and $\boldsymbol{V}_5$ to reduce the full row rank sub-pencil $\boldsymbol{A}_1 - \lambda \boldsymbol{E}_1$ to the following form

$$\boldsymbol{U}_5^T(\boldsymbol{A}_1 - \lambda \boldsymbol{E}_1)\boldsymbol{V}_5 = \begin{bmatrix} \boldsymbol{B}_r & \boldsymbol{A}_r - \lambda \boldsymbol{E}_r & * \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_\infty - \lambda \boldsymbol{E}_\infty \end{bmatrix},$$

where $\boldsymbol{A}_\infty - \lambda \boldsymbol{E}_\infty$ contains the infinity structure of the system pencil $\mathcal{S}(\lambda)$ and the pair $(\boldsymbol{B}_r, \boldsymbol{A}_r - \lambda \boldsymbol{E}_r)$ is controllable with $\boldsymbol{E}_r$ upper-triangular and nonsingular. Compute the final `SPRED` form

$$\widehat{\mathcal{S}}(\lambda) = \operatorname{diag}\{\boldsymbol{U}_5^T, \boldsymbol{I}\}\mathcal{S}_3(\lambda)\operatorname{diag}\{\boldsymbol{V}_5, \boldsymbol{I}\}$$

and $\boldsymbol{Q} \leftarrow \boldsymbol{Q}\operatorname{diag}\{\boldsymbol{U}_5, \boldsymbol{I}\}$, $\boldsymbol{Z} \leftarrow \boldsymbol{Z}\operatorname{diag}\{\boldsymbol{V}_5, \boldsymbol{I}\}$.

Two lower complexity Kronecker-like forms of the system pencil $\mathcal{S}(\lambda)$, the `SRSET` and `SLISEP` forms, are computed at steps 1 and 2 as the pencils $\mathcal{S}_1(\lambda)$ and $\mathcal{S}_2(\lambda)$, respectively. For computing the dual `SRLRED` form the same procedure applied to the pertransposed system pencil $\mathcal{S}(\lambda)^P$ can be used.

For computing the complete orthogonal decomposition at step 1, any rank revealing decomposition of $\boldsymbol{E}$ can be used. The most reliable approach is to compute the singular value decomposition of $\boldsymbol{E}$ and to determine the rank of $\boldsymbol{E}$ on the basis of computed singular values. A less expensive approach is to use the QR-decomposition with column pivoting of $\boldsymbol{E}$. Excepting very special examples (for instance the so-called *Kahan-matrices*), this decomposition has almost the same reliability in determining the rank of a matrix as the singular value decomposition [19]. Thus we decided to use it in implementing the SRSET subroutine in combination with the incremental rank estimation technique proposed in [20]. Reliable software for both decompositions as well as auxiliary routines for the incremental rank estimation, are provided in LAPACK [21]. Notice that in contrast with alternative algorithms [8, 12, 11], a single rank determination is performed involving $\boldsymbol{E}$. In all subsequent computations the preservation of the triangular form and of the full rank structure of intervening "$\boldsymbol{E}$" matrices are crucial for performing the various pencil reductions and for ensuring the $0(n^3)$ computational complexity.

The reductions performed at steps 2 and 4 are based on a reduction technique similar to that introduced in [9] to compute controllability staircase forms of descriptor systems. This technique was used in conjunction with computing system zeros [10] and is described in detail in [2]. The rank determinations are based on QR-decompositions with column pivoting. The main feature of these algorithms is the preservation, during computations of QR-decompositions, of the full rank and of the upper-triangular form of the intervening "$\boldsymbol{E}$" matrices. This feature leads to two important advantages over existing methods. The first advantage is the computational complexity $0(n^3)$. In contrast, the algorithms in [12, 11] have computational complexity $0(n^4)$, because singular value decompositions are used instead of QR-decompositions, and thus the explicit accumulation of left and right transformation matrices is necessary. Notice that the $0(n^4)$ computational complexity is a generic feature of these algorithms and always occurs for example for a randomly generated single-input single-output system. The second advantage arises in comparing the reduction algorithm S-REDUCE of [2] and the improved $0(n^3)$ complexity Algorithm 3.2.1 of [8]. The main weakness of this latter algorithm is the need to update during each QR-like reduction step the rank information on "$\boldsymbol{E}$". This rank updating is in fact equivalent with rank decisions based on QR-decompositions without pivoting and thus it is potentially unreliable. In the S-REDUCE algorithm of [2], "$\boldsymbol{E}$" having always full rank, no such updating is necessary. Instead, two QR-decompositions with column pivoting are necessary to be performed at each step.

The row compression of $\begin{bmatrix} \boldsymbol{B}_c \\ \boldsymbol{D}_c \end{bmatrix}$ at step 3 is performed in two steps. First the rows of $\boldsymbol{D}_c$ are compressed with an orthogonal matrix $\boldsymbol{W}_1$ to an invertible matrix $\boldsymbol{D}_1$ such that

$$\boldsymbol{W}_1^T[\,\boldsymbol{D}_c\;\boldsymbol{C}_c\,] = \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{C}_1 \\ \boldsymbol{O} & \boldsymbol{C}_2 \end{bmatrix}.$$

Then the rows of $\begin{bmatrix} \boldsymbol{B}_c \\ \boldsymbol{D}_1 \end{bmatrix}$ are compressed with an appropriate orthogonal matrix $\boldsymbol{W}_2$ such that

$$\boldsymbol{W}_2^T \begin{bmatrix} \boldsymbol{B}_c & \boldsymbol{A}_c - \lambda \boldsymbol{E}_c \\ \boldsymbol{D}_1 & \boldsymbol{C}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}_i & * \\ \boldsymbol{O} & \boldsymbol{A}_2 - \lambda \boldsymbol{E}_2 \end{bmatrix},$$

where the resulting $\boldsymbol{E}_2$ is upper-triangular and nonsingular. The compression method efficiently combines Givens rotations and row permutations and is described in detail in [8, pages 33-34].

To perform the reductions at step 5 the S-REDUCE algorithm can be used after compressing $\boldsymbol{E}_1$ to a full rank invertible matrix. A more efficient approach is to use the Algorithms 3.3.1 and 3.3.3 proposed in [8] which requires no rank determinations. In computing the `SPRED` form we implemented these two algorithms.

6. Software Implementations

Higher level user callable FORTRAN 77 subroutines have been implemented to compute the Kronecker-like forms described in the previous sections as well as several condensed forms of lower complexity. Some of the lower level subroutines to compute controllability/observability staircase forms of descriptor systems are also of independent interest in some applications. All routines optionally accumulates the left and right orthogonal transformations used during the reductions.

All implementations to compute dual Kronecker-like forms avoid explicit pertransposing by working directly on the original matrices.

The implementations of all routines rely on LAPACK [21] and BLAS calls. The user interface conforms with the implementation standards of the SLICOT library [22]. All routines are extensively commented and *in line* comments serve for documentation purposes. Test programs with files containing test data and test results are available for all user callable routines. Special routines to evaluate the incurred rounding errors are called by all test programs.

7. Conclusions

Several Kronecker-like forms of a system pencil have been introduced and their applications in solving several computational problems have been mentioned. All these Kronecker-like forms can be computed by $0(n^3)$ complexity numerically stable algorithms. The reduction technique can be also applied without modification to the more general case of an arbitrary pencil. A modular collection of LAPACK compatible FORTRAN 77 subroutines to perform the reduction of system pencil to several Kronecker-like forms has been implemented.

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