KYBERNETIKA — VOLUME 26 (1990), NUMBER 2

COMPUTATION OF IRREDUCIBLE GENERALIZED STATE-SPACE REALIZATIONS

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In this paper, an efficient, numerically stable procedure is presented for the computation of irreducible generalized state-space realizations from non-minimal ones. The order reduction is performed by removing successively the uncontrollable and the unobservable parts of the system. Each reduction is accomplished by the same basic algorithm which deflates the uncontrollable part of the system using orthogonal similarity transformations. Applications of the proposed procedure are also presented.

1. INTRODUCTION

Consider the linear time-invariant *generalized state-space model* (GSSM)

$$\begin{aligned} \lambda E\, x(t) &= A\, x(t) + B\, u(t) \\ y(t) &= C\, x(t) \end{aligned} \tag{1}$$

where x, u, and y are the n-dimensional state vector, the m-dimensional input vector, and the p-dimensional output vector, respectively, and where λ is the differential operator $\mathrm{d}/\mathrm{d}t$ for a continuous system or the advance operator z for a discrete system. The matrices E, A, B and C have appropriate dimensions, E and A being square. The system (1) will be referred to alternatively as the triple $\{\lambda E - A, B, C\}$. If the matrix E is singular, the system (1) is also called *singular* or *descriptor* system.

Linear time-invariant systems can also be represented by *differential* or *difference state-space models* (DSSM) of the form

$$\begin{aligned} T(\lambda)\, z(t) &= U(\lambda)\, u(t) \\ y(t) &= V(\lambda)\, z(t) + W(\lambda)\, u(t) \end{aligned} \tag{2}$$

where z is a q-dimensional "internal" state vector, u and y are as above, and $T(\lambda)$, $U(\lambda)$, $V(\lambda)$, $W(\lambda)$ are polynomial matrices having appropriate dimensions with $T(\lambda)$ square. The system (2) will be alternatively denoted by $\{T(\lambda), U(\lambda), V(\lambda), W(\lambda)\}$.

A third frequently used representation of linear constant systems is by the *transfer-*

function matrix model (TFMM)

$$\bar{Y}(\lambda) = G(\lambda)\,\bar{U}(\lambda) \tag{3}$$

where $\bar{U}(\lambda)$ and $\bar{Y}(\lambda)$ are the transforms of the input and output vectors, respectively (the Laplace transform for continuous systems or the Z-transform for discrete systems), and where $G(\lambda)$ is a $p \times m$ rational matrix.

If (1), (2) and (3) correspond to the same system, we have the following basic relations

$$G(\lambda) = C(\lambda E - A)^{-1} B \tag{4}$$

$$G(\lambda) = V(\lambda)\, T^{-1}(\lambda)\, U(\lambda) + W(\lambda) \tag{5}$$

For a given DSSM or TFMM, the determination of a corresponding minimal order (or irreducible) GSSM is known as the *minimal realization problem* (MRP). The MRP has no unique solution. If $\{\lambda E - A, B, C\}$ and $\{\lambda \hat{E} - \hat{A}, \hat{B}, \hat{C}\}$ have the same order and correspond to the same TFMM, then there exist invertible matrices Q and Z such that

$$\lambda \hat{E} - \hat{A} = Q(\lambda E - A)\, Z\,, \quad \hat{B} = QB\,, \quad \hat{C} = CZ \tag{6}$$

Two GSSM will be called *similar* if their matrices are related as in (6) and therefore the transformation (6) will be called *system similarity transformation*. If Q and Z are orthogonal matrices, the transformation will be called *orthogonal system similarity transformation*.

In this paper we describe an efficient, numerically stable procedure for the computation of irreducible GSSM from non-minimal ones. The order reduction is performed by removing successively the uncontrollable and then the unobservable parts of the system. Each reduction step is accomplished by using a new numerically stable algorithm which separates the uncontrollable part of a GSSM. This basic algorithm uses exclusively orthogonal system similarity transformations and is an efficient alternative to existing procedures [1], [2], [3]. The proposed algorithms are presented in Section 2.

The main applications of the new algorithms are: 1) the solution of the MRP; 2) the computation of minimal order inverses of linear systems; and 3) the evaluation of the transfer-function matrices of GSSM. These applications are presented in Section 3. Numerical examples are given in Section 4.

Notations and definitions

Throughout the paper $A \in \mathbb{R}^{m \times n}$ denotes an $m \times n$ matrix with real elements. We use A^{T} for the transpose of A. I or I_n denote identity matrices of known order or of order n, respectively. 0_{mn} denotes an $m \times n$ null matrix. A square matrix Q is orthogonal if $Q^{\mathrm{T}}Q = I$. $\operatorname{im} A$ and $\ker A$ denote, respectively, the image and the kernel of A. For two subspaces $\mathscr{X}$ and $\mathscr{Y}$, $A\mathscr{X}$ is the image of $\mathscr{X}$ under A, and $\mathscr{X} + \mathscr{Y}$ is the sum of subspaces $\mathscr{X}$ and $\mathscr{Y}$. A polynomial matrix is called regular when it is square and has a nonzero determinant. A deflating subspace $\mathscr{X}$ of a regular pencil $\lambda E - A$ satisfies $\dim(A\mathscr{X} + E\mathscr{X}) = \dim \mathscr{X}$, where dim stands for "dimension of". $O(\varepsilon)$ means a quantity of the order of ε.

2. THE IRREDUCIBLE REALIZATION PROCEDURE

In this section we present a numerically stable procedure to compute an irreducible GSSM from a non-minimal one. The order reduction is performed by removing successively the uncontrollable and then the unobservable parts of the system. Each reduction step is accomplished by the same basic procedure which deflates the uncontrollable part of a GSSM using orthogonal system similarity transformations.

The definitions used in this section closely follow the woık of Van Dooren [1]. The controllable and unobservable subspaces of the n-dimensional state-space $\mathscr{X}$ of GSSM $\{\lambda E - A, B, C\}$ can be defined as the deflating subspaces $\mathscr{C}$ and $\bar{\mathscr{O}}$, respectively, which satisfy

$$\mathscr{C} = \inf\{\mathscr{S} \mid \dim(E\mathscr{S} + A\mathscr{S}) = \dim \mathscr{S};\ \operatorname{im} B \subset E\mathscr{S} + A\mathscr{S}\} \tag{7}$$

$$\bar{\mathscr{O}} = \sup\{\mathscr{S} \mid \dim(E\mathscr{S} + A\mathscr{S}) = \dim \mathscr{S};\ \mathscr{S} \in \ker C\}\,. \tag{8}$$

The system is said *controllable* when its controllable subspace $\mathscr{C}$ has dimension n, and *observable* when its unobservable subspace $\bar{\mathscr{O}}$ has zero dimension. We shall assume that the pencil $\lambda E - A$ is regular.

Let r be the dimension of $\mathscr{C}$ defined by (7) and let Z and Q be orthogonal transformation matrices whose first r columns span $\mathscr{C}$ and $E\mathscr{C} + A\mathscr{C}$, respectively. Then we can transform the system $\{\lambda E - A, B, C\}$ as

$$Q^{\mathrm{T}}(\lambda E - A)\,Z = \left[\begin{array}{c|c} \lambda E_c - A_c & * \\ \hline 0 & \lambda E_{\bar{c}} - A_{\bar{c}} \end{array}\right]\begin{matrix}\}r \\ \}n-r\end{matrix}, \quad Q^{\mathrm{T}}B = \left[\begin{array}{c} B_c \\ \hline 0 \end{array}\right]\begin{matrix}\}r \\ \}n-r\end{matrix} \tag{9}$$

with column block widths r and $n-r$,

$$CZ = [\underbrace{C_c}_{r} \mid \underbrace{C_{\bar{c}}}_{n-r}]$$

The reduced order system $\{\lambda E_c - A_c, B_c, C_c\}$ is controllable and has the same TFMM as $\{\lambda E - A, B, C\}$. The eigenvalues of the regular pencils $\lambda E_c - A_c$ and $\lambda E_{\bar{c}} - A_{\bar{c}}$ are called, respectively, the *controllable* and *uncontrollable poles* of the system.

Analogously, let q be the dimension of the unobservable subspace $\bar{\mathscr{O}}$ and let Z and Q be orthogonal transformation matrices whose last q columns span $\bar{\mathscr{O}}$ and $E\bar{\mathscr{O}} + A\bar{\mathscr{O}}$, respectively. Then the system $\{\lambda E - A, B, C\}$ can be transformed to

$$Q^{\mathrm{T}}(\lambda E - A)\,Z = \left[\begin{array}{c|c} \lambda E_o - A_o & 0 \\ \hline * & \lambda E_{\bar{o}} - A_{\bar{o}} \end{array}\right]\begin{matrix}\}n-q \\ \}q\end{matrix}, \quad Q^{\mathrm{T}}B = \left[\begin{array}{c} B_o \\ \hline B_{\bar{o}} \end{array}\right]\begin{matrix}\}n-q \\ \}q\end{matrix} \tag{10}$$

with column block widths $n-q$ and q,

$$CZ = [\underbrace{C_o}_{n-q} \mid \underbrace{0}_{q}]$$

where $\{\lambda E_o - A_o, B_o, C_o\}$ is observable, having the same TFMM as $\{\lambda E - A, B, C\}$.

As can be observed immediately from the form of matrices in (9) and (10), if we have a procedure for computing the controllability form (9) of the system, then the same procedure can be used to determine also the observability form (10) applying it to the dual GSSM $\{\lambda E^T - A^T, C^T, B^T\}$. Therefore, the computation of an irreducible GSSM from a non-minimal one can be performed in two steps: first, determine the controllable part of the system, $\{\lambda E_c - A_c, B_c, C_c\}$ and then determine the observable part $\{\lambda E_{co} - A_{co}, B_{co}, C_{co}\}$ of the resulted controllable part. This system is both controllable and observable, and therefore it is irreducible. It has the same TFMM as the initial GSSM.

2.1 The reduction algorithm

The reduction of the initial system (1) to the form (9) can be accomplished in a numerically stable way using the pencil algorithm of Van Dooren [1]. However, this algorithm, applied as it is stated, is computationally expensive. For example, for a controllable single-input system with E non-singular, the algorithm uses $O(n^4)$ floating-point operations (*flops*). (One flop is roughly equivalent to compute $a + b \times c$, where a, b, c are floating-point numbers.)

Paige [2] outlined a numerically stable algorithm, applicable when E is non-singular, which reduces the system (1) to form (9). This algorithm as well as its modification proposed by Chu [3], performs also $O(n^4)$ flops, the former being a variant of the pencil algorithm for invertible E.

In this section we propose a new numerically stable procedure for the computation of the controllability form (9), which requires only $O(n^3)$ flops. The procedure is applicable regardless E is singular or not. The procedure is based on the following algorithm:

Algorithm 1.

1. Reduce E to *upper-triangular* (U−T) form by using a suitable orthogonal transformation matrix Z_0

$$E \leftarrow EZ_0\,, \quad A \leftarrow AZ_0\,, \quad C \leftarrow CZ_0\,.$$

2. Set $j = 1$, $r = 0$, $n_0 = m$; $E_0 = E$, $A_0 = A$, $B_0 = B$, $Q = I_n$, $Z = Z_0$.
3. Determine the orthogonal transformation matrices Q_j and Z_j to compress the $(n - r) \times n_{j-1}$ matrix B_{j-1} to full row rank while keeping the U-T form of E_{j-1}; perform the transformations and partition the matrices $Q_j^T B_{j-1}$, $Q_j^T E_{j-1} Z_j$ and $Q_j^T A_{j-1} Z_j$ analogously:

$$Q_j^T B_{j-1} \triangleq \underbrace{\left[\begin{array}{c} A_{j,j-1} \\ \hline 0 \end{array}\right]}_{n_{j-1}}\begin{array}{l}\}n_j\\ \}\varrho_j\end{array}; \quad Q_j^T E_{j-1} Z_j \triangleq \left[\begin{array}{c|c} E_{j,j} & E_{j,j+1} \\ \hline 0 & E_j \end{array}\right]\begin{array}{l}\}n_j\\ \}\varrho_j\end{array}$$

with the column blocks of $Q_j^T E_{j-1} Z_j$ of widths n_j and ϱ_j;

$$Q_j^T A_{j-1} Z_j \triangleq \left[\begin{array}{c|c} A_{j,j} & A_{j,j+1} \\ \hline B_j & A_j \end{array}\right]\begin{array}{l}\}n_j\\ \}\varrho_j\end{array}$$

with column blocks of widths n_j and ϱ_j.