A New Architecture for Optimization Modeling Frameworks

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Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b, \end{array}$$

with variable $x \in \mathbf{R}^n$

- objective and inequality constraints $f_0, \ldots, f_m$ are convex

  for all x, y, $\theta \in [0, 1]$,

  $$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

  i.e., graphs of f_i curve upward
- equality constraints are linear

Why convex optimization?

- beautiful, fairly complete, and useful theory
- solution algorithms that work well in theory and practice
- **many applications** in
  - machine learning, statistics
  - control
  - signal, image processing
  - networking
  - engineering design
  - finance

. . . and many more

How do you solve a convex problem?

- use someone else's ('standard') solver (LP, QP, SOCP, ...)
  - easy, but your problem **must** be in a standard form
  - cost of solver development amortized across many users
- write your own (custom) solver
  - lots of work, but can take advantage of special structure
- use a convex modeling language
  - transforms user-friendly format into solver-friendly standard form
  - extends reach of problems solvable by standard solvers

Convex modeling languages

- long tradition of modeling languages for optimization
  - AMPL, GAMS
- modeling languages for convex optimization
  - CVX, YALMIP, CVXGEN, CVXPY, Convex.jl, RCVX
- function of a convex modeling language:
  - check/verify problem convexity
  - convert to standard form

Disciplined convex programming (DCP)

- system for constructing expressions with known curvature
  - constant, affine, convex, concave
- expressions formed from
  - variables
  - constants and parameters
  - library of functions with known curvature, monotonicity, sign
- basis of all convex modeling systems
- more at dcp.stanford.edu

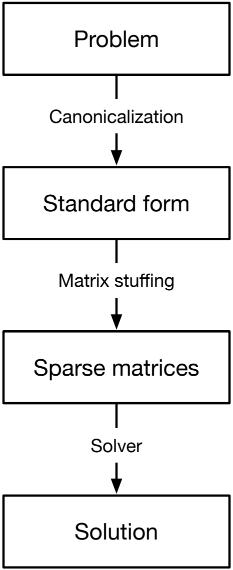
The one rule that DCP is based on

$h(f_1(x), \ldots, f_k(x))$ is convex when h is convex and for each i

- h is increasing in argument i, and f_i is convex, or
- h is decreasing in argument i, and f_i is concave, or
- f_i is affine

- there's a similar rule for concave compositions
(just swap convex and concave above)

Traditional architecture for optimization frameworks



Standard (conic) form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \in \mathcal{K} \end{array}$$

with variable $x \in \mathbf{R}^n$

- $\mathcal{K}$ is convex cone
  - $x \in \mathcal{K}$ is a generalized nonnegativity constraint
- linear objective, equality constraints
- special cases:
  - $\mathcal{K} = \mathbf{R}^n_+$: linear program (LP)
  - $\mathcal{K} = \mathbf{S}^n_+$: semidefinite program (SDP)
- general interface for solvers

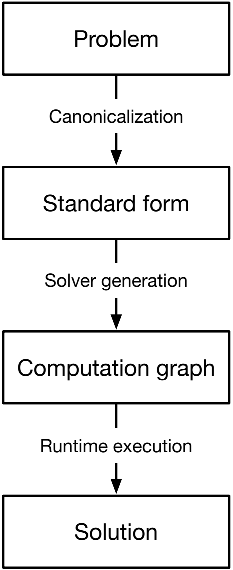
Traditional cone solvers

- CVXOPT (Vandenberghe, Dahl, Andersen)
  - interior-point method
  - Python
- ECOS (Domahidi)
  - interior-point method
  - supports exponential cone
  - compact, library-free C code
- SCS (O'Donoghue)
  - first-order method
  - parallelism with OpenMP
  - GPU support
- others: GLPK, MOSEK, GUROBI, Cbc, Elemental, ...
- traditional architecture has been enormously successful
  - solvers based on interior point methods highly robust
  - solvers portable to new platforms with linear algebra libraries
    - BLAS, LAPACK, SuiteSparse, *etc.*

Drawbacks of traditional architecture

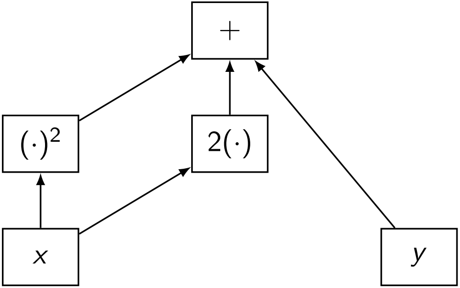
- for large problems, direct solutions to linear systems involving the A matrix can be very expensive
- first-order methods (SCS) allow the use of indirect methods for linear solver subroutine
- but, representing all linear operators as sparse matrices can be inefficient
  - *e.g.*, FFT-based convolution
- also, (most) existing solvers do not take advantage of modern platforms, *e.g.*, GPUs, distributed

Graph-based architecture



Computation graphs

- computation graph for $f(x, y) = x^2 + 2x + y$



- simple transformations produce computation graphs for function gradient and adjoint
  - key operations in first-order and indirect solvers

Computation graph frameworks

- huge momentum and engineering effort from deep learning community
  - TensorFlow, Theano, Caffe, Torch, ...
- support a wide variety of computational environments
  - CPU, GPU, distributed clusters, phones, ...
- given a computation graph, existing frameworks implement gradient descent
- for optimization, first-order and indirect solvers fit naturally
- limited support for sparse matrix factorizations, which are required by interior point methods, direct solvers

Generating solver graphs

- solver generation implemented with functions parameterized by graphs or graph generators
- *e.g.*, conjugate gradient for solving linear system $Ax = b$

```
def cg_solve(A, b, x_init, tol=1e-8):
  delta = tol*norm(b)
  def body(x, k, r_norm_sq, r, p):
    Ap = A(p)
    alpha = r_norm_sq / dot(p, Ap)
    x = x + alpha*p
    r = r - alpha*Ap
    r_norm_sq_prev = r_norm_sq
    r_norm_sq = dot(r,r)
    beta = r_norm_sq / r_norm_sq_prev
    p = r + beta*p
    return (x, k+1, r_norm_sq, r, p)
  def cond(x, k, r_norm_sq, r, p):
    return tf.sqrt(r_norm_sq) > delta
  r = b - A(x_init)
  loop_vars = (x_init, tf.constant(0), dot(r, r), r, r)
  return tf.while_loop(cond, body, loop_vars)[:3]
```

Software implementation and numerical examples

- based on CVXPY, a convex optimization modeling framework
- solves convex problems using TensorFlow
- implements a variant of SCS, a first-order method
- linear subproblems solved with conjugate gradient
- experiment platform details
    - 32-core Intel Xeon 2.2Ghz processor
    - nVidia Titan X GPU with 12GB RAM

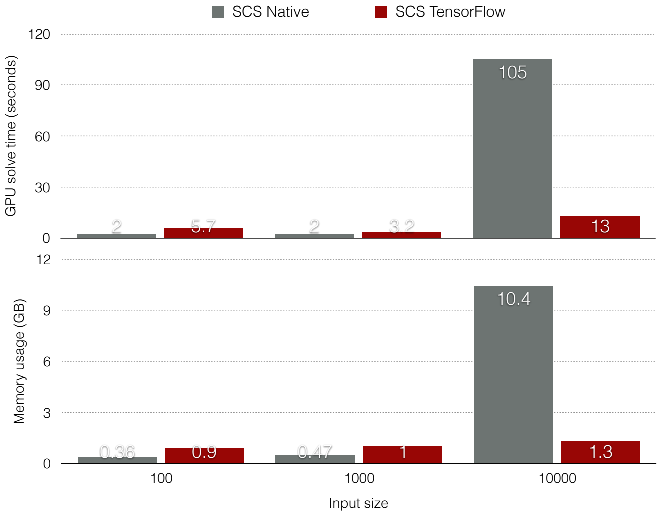
Nonnegative deconvolution example

$$\begin{array}{ll} \text{minimize} & \|c * x - b\|_2 \\ \text{subject to} & x \geq 0, \end{array}$$

with variable $x \in \mathbf{R}^n$, problem data $c \in \mathbf{R}^n$, $b \in \mathbf{R}^{2n-1}$

```
from cvxpy import *
from cvxflow import scs_tf
x = Variable(n)
f = norm(conv(c, x) - b, 2)
prob = Problem(Minimize(f), [x >= 0])
scs_tf.solve(prob)
```

Comparison on nonnegative deconvolution



Conclusions

- convex optimization is useful
- convex modeling languages make it easy
- graph-based architectures help it scale
- open source Python libraries available
  - cvxpy: `cvxpy.org`
  - cvxflow: `github.com/cvxgrp/cvxflow`

More details for nonnegative deconvolution

	small	**medium**	**large**
variables n	101	1001	10001
constraints m	300	3000	30000
nonzeros in A	9401	816001	69220001
SCS native			
solve time, CPU	0.1 secs	2.2 secs	260 secs
solve time, GPU	2.0 secs	2.0 secs	105 secs
matrix build time	0.01 secs	0.6 secs	52 secs
memory usage	360 MB	470 MB	10.4 GB
objective	1.38×10^0	4.57×10^0	1.41×10^1
SCS iterations	380	100	160
avg. CG iterations	8.44	2.95	3.01
SCS TensorFlow			
solve time, CPU	3.4 secs	5.7 secs	88 secs
solve time, GPU	5.7 secs	3.2 secs	13 secs
graph build time	0.8 secs	0.8 secs	0.9 secs
memory usage	895 MB	984 MB	1.3 GB
objective	1.38×10^0	4.57×10^0	1.41×10^1
SCS iterations	480	100	160
avg. CG iterations	2.75	2.00	2.00